Math 3341 **Nonlinear Dynamics and Chaos with Applications** Spring 2015

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Office Hours: M: 6 – 7 and W: 4:45 – 5:45

Text: "Nonlinear Dynamics and Chaos", Steven H. Strogatz (2nd edition, Publisher Westview)
Meeting times and location: MW 2:50 pm - 4:30 pm, Shillman Hall 210

Prerequisites: *Very little.* Single variable calculus, separable differential equations, basic linear algebra. I will make the course as self-contained as possible.

Grading: TBD

Class Schedule & Homework List

Please note that the schedule is very **tentative** and may be changed at any point. Students are responsible for coming to class and if absent, students still need to be responsible for all material covered and changes announced in class. It is the students' responsibility to check emails and Blackboard.

	Section	Topic	Assignment	
Jan. 12 – 16	1 – 2.3	Review basics of differential equations Overview and 1-dimensional flows		
Mon. Jan 19		**Martin Luther King Birthday, no class**		
Jan. 20 - 23	3.0 - 3.4	Dimensional bifurcations		
Jan. 26 - 30		*Review, Discussion, Tests*		
Feb. 2-6	4 .0 - 4.4	Flows on the circle		
Feb. 9-13	5.1 – 5.3	Linear Systems		
Feb. 17-20	6.0 – 6.3	Linear Systems and Linearization		
Feb. 2-27	6.4 – 6.6	Continued . . .		
Mar. 2-6		*Review, Discussion, Tests*		
Mar. 9-13		**Spring Break, No Class (have a blast)**		
Mar. 16 - 29	8.0 – 8.4	2-dimensional bifurcations		
Mar. 30 -Apr. 3				
Apr. 6-10				
Apr. 13 - 17				
Mon. Apr. 20		**Patriot's day, no class**		
April 21-22		Review		

Dynamic Systems — Introduction and Review 2015/01/12

Textbook: Nonlinear Dynamics and Chaos
Steven H. Strogatz

Review of Differential Equations, First Order

$$\frac{dy(x)}{dx} = y(x) + c$$

$x \rightarrow$ independent variable
$y \rightarrow$ dependent variable

ODE

PDE: more than one ind. var.

$$\frac{\partial y(x,t)}{\partial x} - 5\frac{\partial y(x,t)}{\partial t} + y(x,t) = 0$$

Linearity of DEs:

A DE is linear iff the dependent variable is linear.

$$\frac{dy(x)}{dx} + x^2 - y(x) = \sin(x) \longrightarrow \text{Linear}$$

Nonlinear DEs contain variable products or nonlinear functions of the dependent variable, eg:

$$y^2,\ y\frac{\partial y}{\partial t},\ \left(\frac{\partial y}{\partial t}\right)^3,\ \sin y,\ \text{etc.}$$

Coupled Differential Equations:
More than one dep. var

$$\frac{d^2 x_1(t)}{dt^2} = x_1(t) - x_2(t)\ ;\ \frac{d^2 x_2(t)}{dt^2} = x_2(t) - x_1(t)$$

Homogeneity of DEs:

$$\frac{d^3y(x)}{dx^3} + \frac{dy(x)}{dx} - y(x) = 0 \longrightarrow \text{Homogeneous}$$

↳ Every term in the equation has the dependent variable. (No constants)

Can be written as the product of $y(x)$ and a term, as in $y(x)\left[\frac{\partial^3}{\partial x^3} + \frac{\partial}{\partial x} - 1\right] = 0$

Always produces the trivial solution, $y(x) = 0$, an undriven solution in which the system is not acted upon by external force.

Superposition Principle:

If you have solutions $y_1(x)$ and $y_2(x)$, when linear and homogeneous, $y(x) = A \cdot y_1(x) + B \cdot y_2(x)$

Solutions to Nonhomogeneous DE are of form:

$$y(x) = y_h(x) + y_p(x)$$

homogeneous solution

Methods of Solving 1st Order DEs:

1. Seperation of Variables
2. formation of Exact Differential
3. Integrating factor

Seperation of Variables:

If $\frac{\partial y(x)}{dx} = -\frac{P(x)}{Q(y)} \longrightarrow Q(y) \cdot dy = -P(x) \cdot dx$

2015/01/12

Ex | $\frac{dy}{dx} = \frac{\cos(x)}{y}, \quad y(0)=0$

$$\int y\,dy = \int \cos(x)\,dx \rightarrow \frac{1}{2}y^2 = \sin(x) + c_1$$

$$\xrightarrow{y(0)=0} 0 = 0 + c_1 \rightarrow c_1 = 0 \therefore y(x) = \pm\sqrt{2\sin(x)}$$

Ex | $y' = x + xy^2, \quad y(0)=1$

$$\int \frac{dy}{(1+y^2)} = \int x\,dx \rightarrow \tan^{-1} y = \frac{1}{2}x^2 + c_1$$

$$\xrightarrow{y(0)=1} \frac{\pi}{4} = c_1 \therefore y(x) = \tan\left(\frac{1}{2}x^2 + \frac{\pi}{4}\right)$$

Formation of Exact Differentials

$$\frac{dy(x)}{dx} = -\frac{R(x,y)}{S(x,y)} \Rightarrow R(x,y)\,dx + S(x,y)\,dy = 0$$

find $\varphi(x,y)$ | $\begin{cases} \frac{\partial \varphi(x,y)}{\partial x} = R(x,y) & ① \\ \frac{\partial \varphi(x,y)}{\partial y} = S(x,y) & ② \end{cases}$

Because $\frac{\partial \varphi(x,y)}{\partial x} dx + \frac{\partial \varphi(x,y)}{\partial y} dy = 0, \quad \varphi = C$

Ex | $\frac{dy}{dx} = -\frac{3x^2 + 2y^2}{4xy}$ ← $R(x,y)$ ← $S(x,y)$

$$(3x^2 + 2y^2)\,dx + (4xy)\,dy = 0$$

$\frac{\partial \varphi}{\partial x} = 3x^2 + 2y^2$	$\frac{\partial \varphi(x,y)}{\partial y} = 4xy$
$\varphi(x,y) = x^3 + 2y^2x + C_1(y)$	$\varphi(x,y) = 2xy^2 + c_2(x)$

$$\varphi(x,y) = x^3 + 2y^2x + C_0 \rightarrow \xrightarrow{y(1)=1} \varphi(1,1) = 3 = C$$

$$x^3 + 2y^2x = 3$$

$$2y^2x = \frac{3-x^3}{2x}; \quad y = \pm\sqrt{\frac{3-x^3}{2x}}$$

Using an Integrating factor

$$\frac{dy(x)}{dx} + P(x) \cdot y(x) = G(x) \Big\} \; P(x), G(x) \rightsquigarrow \text{independent variable}$$

$$\alpha(x) \frac{dy(x)}{dx} + \alpha(x) P(x) y(x) = \alpha(x) G(x) \rightarrow \frac{\partial}{\partial x}\left[\alpha(x) \cdot y(x)\right]$$

$$\star \quad \cdots \boxed{\alpha(x) = e^{\int P(x)\,dx}} \quad \text{so} \quad y_g(x) = \frac{\int \alpha(x) G(x)\,dx + C_0}{\alpha(x)}$$

Ex] $\quad x \frac{dy}{dx} = x^2 + 3y \;;\; x > 0$

$$\frac{dy}{dx} = x + 3x^{-1}y \rightarrow \frac{dy}{dx} \underbrace{-3x^{-1}}_{P(x)} y = \underbrace{x}_{G(x)}$$

$$\alpha(x) = e^{\int P(x)dx} = e^{-3\ln x} = x^{-3}$$

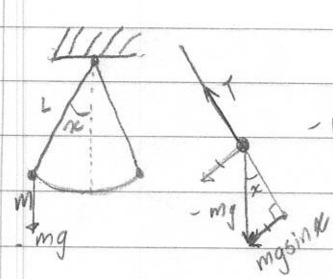
$$x^{-3}\frac{dy}{dx} - 3x^{-4}y = x^{-2}$$

$$\int \frac{dy}{dx}\left(\frac{1}{x^3}y\right) = \int \frac{1}{x^2} \rightarrow \frac{1}{x^3}y = -x^{-1}$$

§2.1 Introduction to Dynamical Systems 2015/01/14

Roots in classical mechanics: Newton/Liebniz in the 1600s
Lagrange, Hamilton advanced field in 1800s.
Classical E&M applications in 1800's: Faraday, Maxwell, etc
Einstein unifies these into Special Relativity (1905) → General Relativity
1900's (1928) Quantum Mechanics pioneered by Dirac, Pauli, Heisenberg, Shrodinger, Bohr, etc)
Later unified to Quantum Field Theory (QM & Special Relativity)
Final goal is unification of General Relativity & Quantum Field Theory
Quantum Mech introduced probablistic methodology into mechanics
H. Poincaré: late 1800s, early 1900s. Showed that 3-body problem impossible to solve
— chaotic systems (4 dimensions needed for chaos)
— geometrical system for dynamics
1920's - 1950's — Birkhoff, Kolmogrov, Arnold, Mose
1963, Lorenz comes around, applies to computer (strange attractors)
1970s, nonlinear systems are underpinning for turbulence, universality, phase transitions. KCT biological systems showing nonlinear oscillation (Winfree). Mandelbrot & fractals

Mathematical models for dynamic systems
① Differential equations : Continuous in time
② Difference equations, iterated maps : Discrete in time



$$f = ma$$

$$-mg \sin x = m(\ddot{Lx}) = mL\ddot{x}$$

$$\ddot{x} = -\frac{g}{L}\sin x$$

$$\ddot{x} + gL^{-1}\sin x = 0$$

$x(t) \rightarrow$ Elliptical function

$$\boxed{\begin{array}{l} \ddot{y} + k^2 y = 0 \\ y = c_1 \sin(kt) + c_2\cos(kt) \end{array}}$$

$\theta' =$

(remember the approximation $\sin x \approx x$)

$$\ddot{x} + gL^{-1}x \approx 0 \leftarrow \text{linear}$$

$\ddot{\theta}L = mg\sin\theta$

$$\ddot{x} + \omega_0^2 x \approx 0 \quad \text{where } \omega_0 = \sqrt{\frac{g}{L}}$$

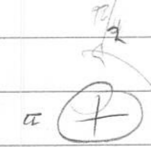
$$x(t) \approx c_1\sin(\omega t) + c_2\cos(\omega t)$$

if $x_{max} < 15°$, approximately

Nonlinearity by Example

Ex | $\dot{x} = \sin x$

Questions we can ask:
What happens at $\lim t \to \infty$?
What if $x(t=0) = \pi/4$?
How does $x(t=0)$ influence the function?

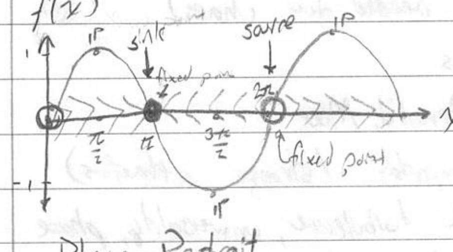


Solution: $\frac{dx}{dt} = \sin x$

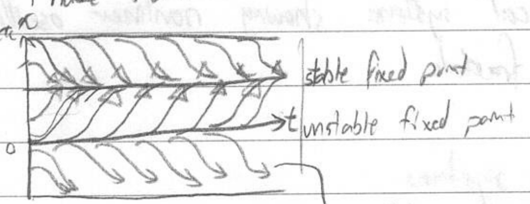
$$\int \frac{dx}{\sin x} = \int dt$$

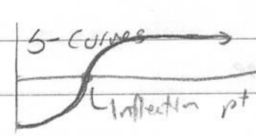
$$-\ln|\csc x + \cot x| = t + c$$

Suppose $\dot{x} = f(x)$, $f(x) = \sin x$



Phase Portrait

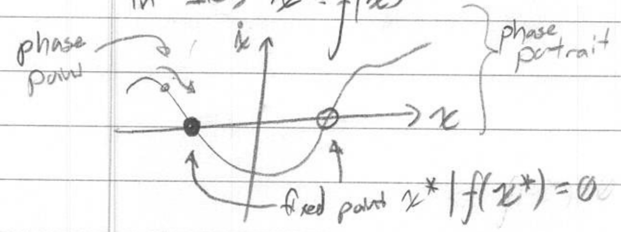




§2.2 Fixed Points & Stability

2015/01/14

In 1D, $\dot{x} = f(x)$

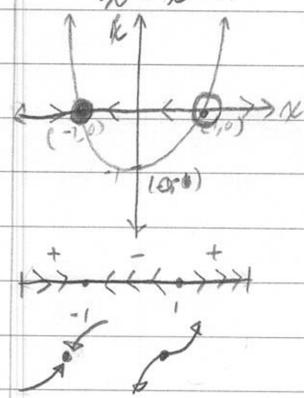


fixed points
- stable: sufficiently small disturbances are attracted
- unstable: sufficiently small disturbances are repelled

trajectory: $x(t)$ (depends on x_0)

Ex find all fixed points and classify by stability

$\dot{x} = x^2 - 1$

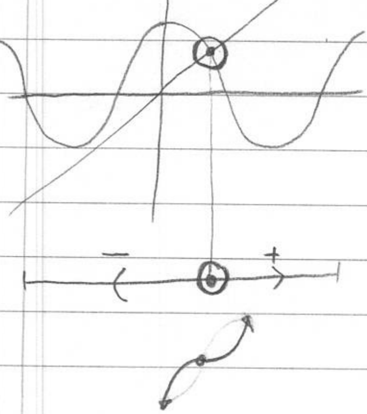


$x^* = \{-1, 1\}$

$x^* = -1: \ddot{x} = 2x\big|_{x=-1} = -2 \longrightarrow$ Stable (Sink)

$x^* = 1: \ddot{x} = 2x\big|_{x=1} = 2 \longrightarrow$ Unstable (Source)

Ex $\dot{x} = x - \cos x$; $0 = x - \cos x$; $x = \cos x$



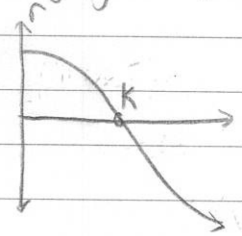
Unstable fixed point at $x^* = \cos x^*$

§2.3 Population Growth 2015/01/21

Let $N(t)$ be the population at t

Exponential growth: $\dot{N}(t) = rN(t) \longrightarrow N(t) = N_0 e^{rt}$; $N_0 = N(t=0)$
$r>0 \Rightarrow$ Exponential growth ; $r<0 \rightarrow$ Exponential Decay

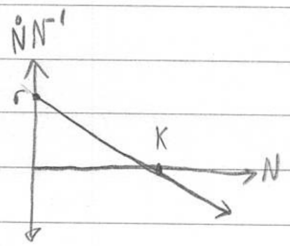
However, these are idealized. Emperically, populations have carrying capacities dependent on resources, predation.



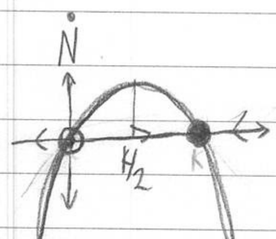
Varhurst, 1838

$\dot{N} = f(N(t), t)$

$r = \frac{\dot{N}}{N}$; In the Varhurst model, $\frac{\dot{N}}{N} = r\left(1 - \frac{N}{K}\right)$ ← nonlinear

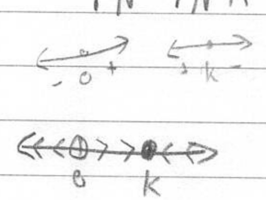


What are the fixed points?
What are their stabilities?
What happens to N_0 as $t \to \infty$



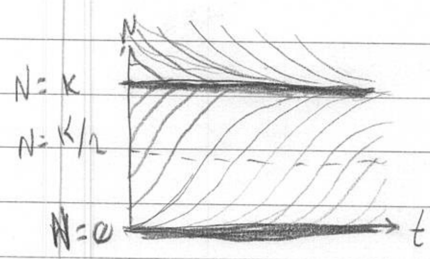
$\dot{N} = rN(1-NK^{-1})$
$= rN - rN^2K^{-1}$

$NK^{-1} = 1$; $N = K$



CRITIQUE OF VARHURST MODEL

- works well w/ simple organisms: microbes, yeast, etc
- ineffective at describing more complex biological systems: eggs, lifecycles, mating, etc



$t \to \infty : N \to K$
for $N_0 > 0$;
$t \to \infty : N \to 0$
for $N_0 = 0$

§2.4 Linear Stability Analysis (Perturbation theory)

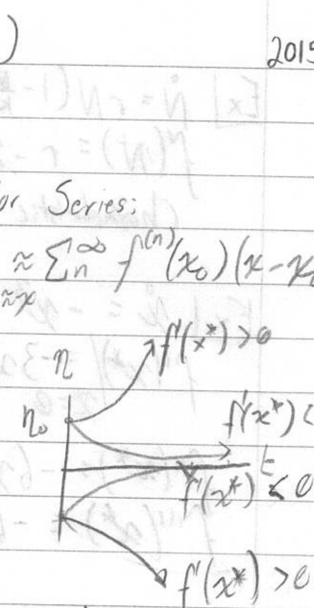
2015/01/21

Suppose $\dot{x} = f(x)$.

$\eta \equiv x - x^*$, where η is very small.
$\quad \hookrightarrow$ constant

Taylor Series:
$$f(x)\Big|_{x_0 \approx x} \approx \sum_n^\infty f^{(n)}(x_0)(x-x_0)^n (n!)^{-1}$$

$$\dot{\eta} = \dot{x} = f(\eta + x^*)$$
$\downarrow$ Taylor
$$\dot{\eta}\Big|_{x^*} = f(x^*) + \eta f'(x^*) + O(\eta^2) \rightarrow \dot{\eta} \approx \eta \underbrace{f'(x^*)}_{\text{constant}}$$



Let $r = f'(x^*)$; $\displaystyle\int \frac{d\eta}{\eta} = \int r\,dt$; $\underbrace{\eta(t) = \eta_0 e^{f'(x^*)t}}$

Perturbation increases when $f'(x^*) > 0$,
decreases when $f'(x^*) < 0$.

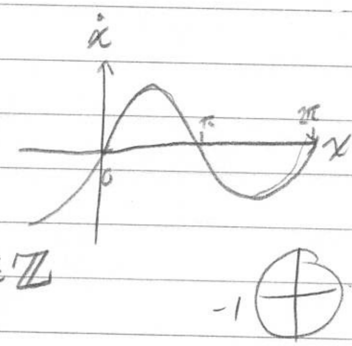
That is, the function is stable when $f'(x^*) < 0$, unstable when $f'(x^*) > 0$. When $f'(x^*) = 0$, consider η^2 or greater terms.

The magnitude of $f'(x^*)$ has to do with convergence speed. We can use this to determine relative stability through doubling times, etc.

$\dfrac{1}{|f'(x^*)|} \leftarrow$ seconds $\qquad \underset{\uparrow \text{units}}{\eta} = \underset{\uparrow \text{units}}{\eta_0}\, e^{\overset{\text{Hertz}}{f'(x^*)}\, \overset{\text{seconds}}{t}}$

Characteristic time (e.g., to grow/decay by a factor of e):
$|f'(x^*)|^{-1}$

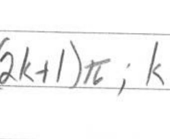
Ex] $\dot{x} = \sin x$



x^*: $0 = \sin x$; $x = k\pi$, $k \in \mathbb{Z}$

$f'(x^*) = \cos x$; $f'(k\pi) = \cos(k\pi)$

$f'(x^*) = 1$ for $x^* = (2k)\pi$
$\qquad\;\; = -1$ for $x^* = (2k+1)\pi$ for $k \in \mathbb{Z}$



$\dot{x} = \sin x$; stable for $x^* = 2k\pi$; unstable for $x^* = (2k+1)\pi$; $k \in \mathbb{Z}$
Characteristic time $= 1$, for all fixed points.

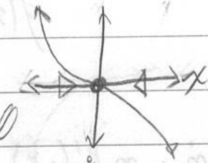
Ex $\dot{N} = rN\left(1-\frac{N}{K}\right)$, $N^* = \{0, K\}$; For $N^* = 0$: $f'(0) = r$: unstable

$f'(N) = r - 2rNK^{-1}$ $\qquad$ $N^* = K$: $f'(K) = -r$: stable

Characteristic time: r^{-1} for all N^* $\qquad$ ($f'(x^*)$ indicates convergence in nits or natural decibells ≈ w/e)

Ex $\dot{x} = -x^3$

$f'(x^*)\big|_{x^*=0} = -3x^2 = 0$



$f''(x^*) = -6x = 0$

$f'''(x^*) = -6 \Rightarrow$ stable

Ex $\dot{x} = x^2$; $x^* = 0$ $\qquad$ semi-stable, actually.

$f'(x^*) = 2x = 0$

find the $\mathcal{O}(\eta^2)$ term, don't cheat

§2.8 Numerical Calculation — Euler's Method

2015/01/26

$\dot{x} = f(x) \rightarrow$ hard to integrate $\frac{dx}{f(x)}$

Goal: given $X(t_0) = x_0$, find $X(t)$

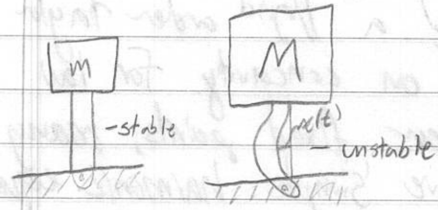
$x(t + \Delta t) = X(t) + \dot{x}(t)\Delta t \leftarrow$ 1st Order Difference Equation

Error is $O(\Delta t)$

Runge-Kutta — $O(\Delta t^4)$ — 4 simultaneous equations

§ 3.1 Bifurcation Theory - Saddle Node Bifurcation

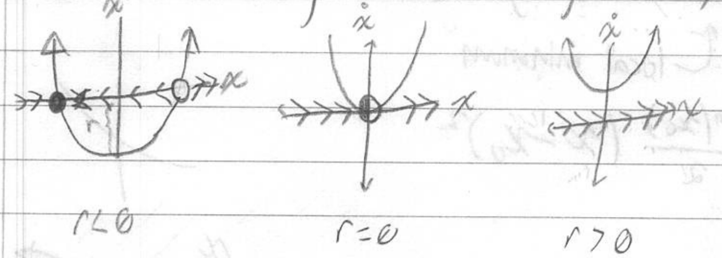
2015/01/26



A beam subjected to weight buckles not proportionally, but avalanches over a threshold.

Feigenbaum
Unimodal Map
Period Doubling
"Brain Dynamics"

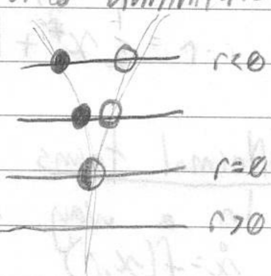
Ex] $\dot{x} = r + x^2 \equiv f(x)$; x is dynamical, r is a parameter.



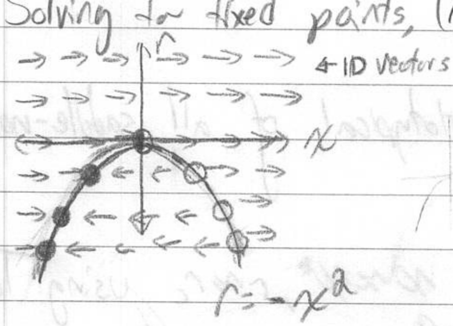
Saddle-node bifurcation uses the parameter r to create/destroy fixed points

two fixed points - stable and unstable | one fixed pt: half-stable | no fixed points

As r increases, fixed points approach, collide, and annihilate.



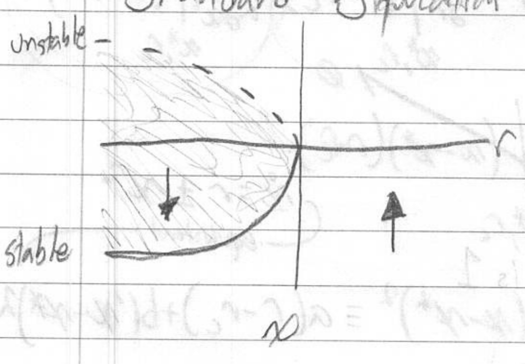
Solving for fixed points, $(x^*)^2 = -r$



Also called:

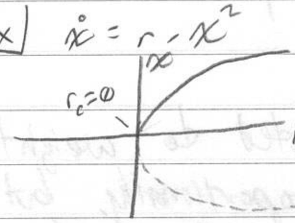
- fold bifurcation,
- turning-point bifurcation
- blue sky bifurcation ← Abraham + Shaw (1958)

Standard Bifurcation plot



Bifurcation Point Annihilation!

Ex $\dot{x} = r - x^2$



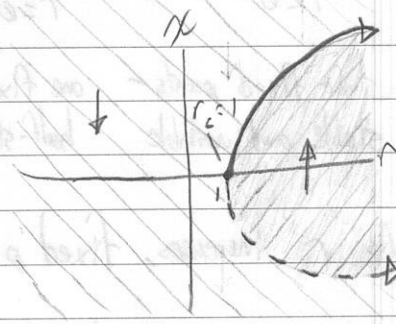
$r \pm x^2$ is often sufficient as it acts like a 2nd order Taylor expansion on concavity for low energies near fixed points, many systems give simple harmonic equations
$\ddot{x} + k^2 x = 0$

$$f(x)\Big|_{x \approx x_0} = f(x_0) + \underbrace{\cancel{f'(x_0)}}_{\text{local minimum}}^{0}(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \mathcal{O}(x - x_0)^3$$

$$f(x)\Big|_{x \approx x_0} \approx f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

Ex $\dot{x} = r - x - e^{-x}$

$$= r - x - \left[1 - x + \frac{x^2}{2} + \mathcal{O}(x^3)\right]$$
$$= \boxed{(r-1) - \frac{1}{2}x^2} + \mathcal{O}(x^3)$$
$$\approx (r-1) - \frac{1}{2}x^2$$
$$r = \frac{1}{2}x^{*2} + 1 \longrightarrow$$



Normal forms

In a way, $\dot{x} = r \pm x^2$ is prototypical of all saddle-node bif's.

$\dot{x} = f(x, r)$

Goal: examine systems where $x \approx x^*$, $r \approx r_c$ using Taylor

$$\dot{x} = f(x,r)\Big|_{\substack{x \approx x^* \\ r \approx r_c}} \approx \cancel{f(x^*, r_c)}^{0} + \cancel{\frac{\partial f}{\partial x}\Big|_{x^*, r_c}(x - x^*)}^{0} + \frac{\partial f}{\partial r}\Big|_{x^*, r_c}(r - r_c) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x^*, r_c}(x - x^*)^2$$

$$+ \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial r^2}\Big|_{x^*, r_c}(r - r_c)^2}^{\approx 0} + \cancel{\frac{\partial^2 f}{\partial x \partial r}\Big|_{x^*, r_c}(x - x^*)(r - r_c)}^{0}$$

by simplification note coefficient is 1

$\dot{x} = r \pm x^2$ equivalent

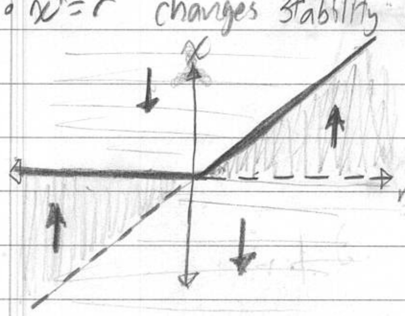
$$\dot{x} = f(x,r)\Big|_{\substack{x \approx x^* \\ r \approx r_c}} \approx \underbrace{\frac{\partial f}{\partial r}\Big|_{x^*, r_c}}_{\equiv a}(r - r_c) + \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x^*, r_c}}_{\equiv b}(x - x^*)^2 \equiv a(r - r_c) + b(x - x^*)^2$$

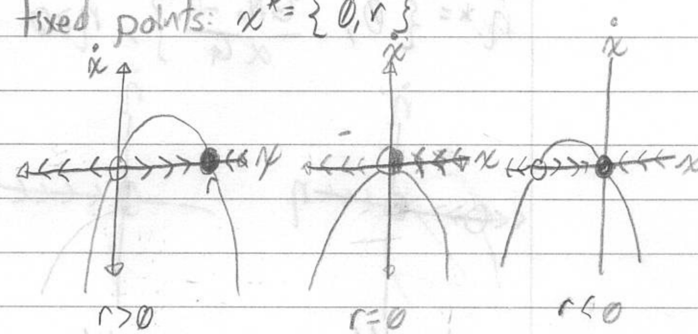
§3.2 Transcritical Bifurcation

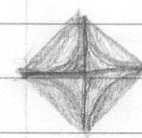
2015/01/26

Normal form: $\dot{x} = rx - x^2$ Fixed points: $x^* = \{0, r\}$

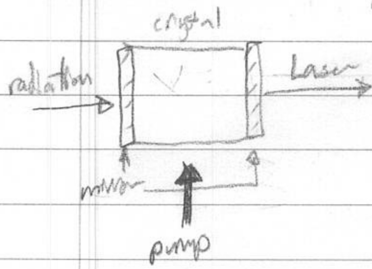
Transcritical:
- No annihilation
- $x^* = r$ changes stability at bif. pt.







Ex) LASER dynamics



A threshold of pumping energy is reached when the crystal gives out laser light.

Friday:
567 Lake
Homework

friday afternoons?

Let $\eta(t)$ be the quantity of photons in the laser

$\dot{\eta}$ = gain - loss = $G\eta N - k\eta$

rate of photon loss

number of atoms in excited state

medium gain, $G > 0$

$N(t) = N_0 - \alpha\eta$

fraction of responding atoms

steady-state excitement

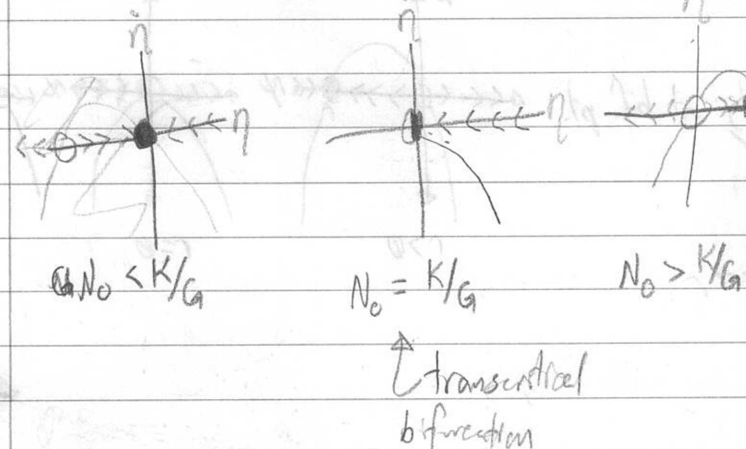
$$\begin{aligned}\dot{\eta} &= G\eta(N_0 - \alpha\eta) - k\eta \\ &= -\alpha G\eta^2 + \eta(GN_0 - k) \\ &= \eta[-\alpha G\eta + GN_0 - k]\end{aligned}$$

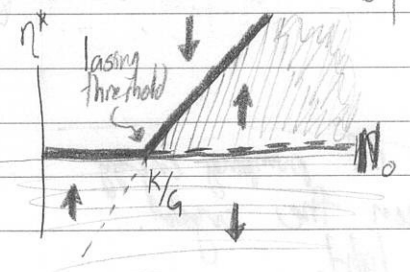
$\dot{\eta} = \eta[-\alpha G \eta + G N_0 - k]$

$\eta^* = \{ 0, \frac{G N_0 - k}{\alpha G} \}, \ \eta \geq 0$

~~Lasing~~

Critical: $N_0 > K/G$





§3.4 Pitchfork Bifurcation

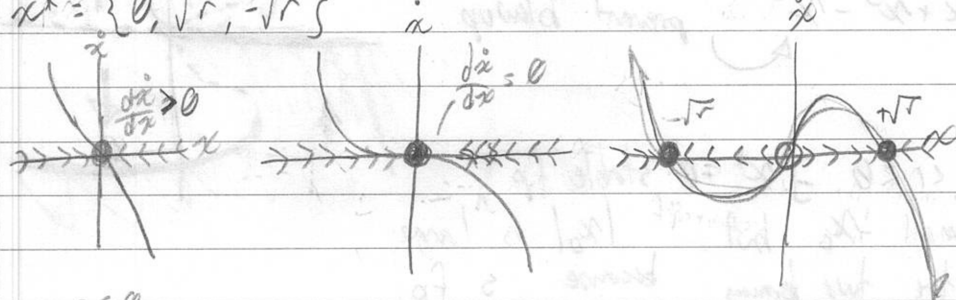
When $\exists$ an underlying symmetry in the system

☆ Emma Nöther — Conservation laws come from continuous symmetry

Two varieties — super-critical, subcritical

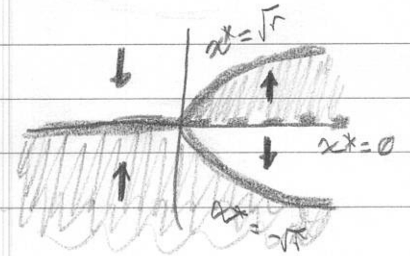
$\boxed{\dot{x} = rx - x^3}$ ← Characteristic equation of supercritical pfbf

$x^* = \{0, \sqrt{r}, -\sqrt{r}\}$

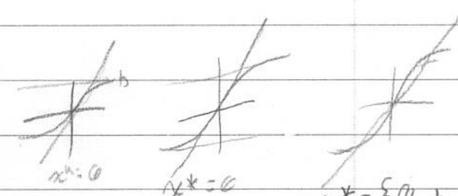


$r < 0$ strong f.p. (exp) $\quad r = 0$ weak f.p. (algebraic) $\quad r > 0$

$x^* = 0$: s → u as $r: 0 \to +$

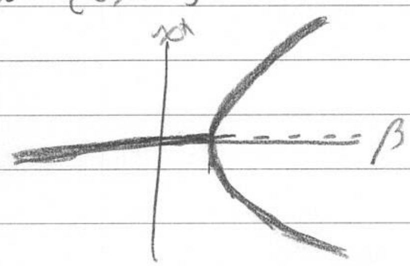


Pitchfork Bifurcation



Ex $\dot{x} = -x + \beta \tanh x$; $\beta \tanh x = x$

$x^* = \{0, \pm \sim\}$ $\qquad \beta = x \tanh^{-1} x$

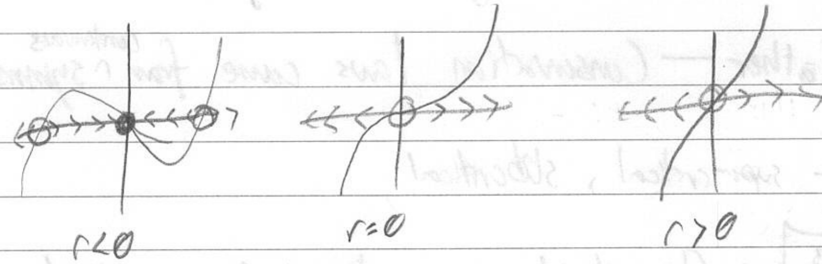


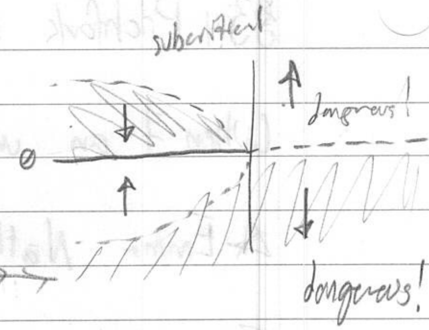
$$x_{n+1} = x_n - (\Delta t) x_n + (\Delta t) \beta \tanh x_n$$

$$x_{n+1} = (1 - \Delta t) x_n + (\Delta t) \beta \tanh x_n$$

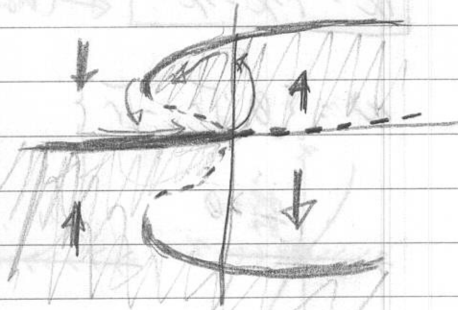
Subcritical P.f. Bifurcation — $\dot{x} = rx + x^3$

$x^* = \{0, \pm\sqrt{r}\}$



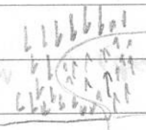


$\dot{x} = rx + x^3 - x^5$ prevent blowup



In $r_s < r < 0$ $\exists$ $x^* = 0$ stable f.p. to small x_0 but if $|x_0|$ is large then the two branches become s f.p.

Origin locally stable, not globally.

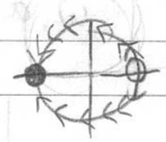


§ 4.1 - 4.3 Flows on a Circle

$\dot\theta = f(\theta)$ — Vector field on a circle
Similar to linear, but periodic.

§ 4.1

$\dot\theta = \sin\theta$, $\theta^* = \{0, \pi\}$



$\dot\theta = \theta$ $\quad$ $\theta^* = 0$

← Not unique.

To be a genuine circular vector field, must be periodic over 2π. (endpoints on π and $-\pi$ must be equal)

A vector field on a circle is a rule that assigns a unique angular velocity vector to each point on the circle. $\dot\theta = f(\theta)$ for $f(\theta) = f(\theta + 2\pi)$

Ex | Uniform Oscillation: $\dot\theta = \omega$; $\theta(t) = \omega t + \theta_0$

Every $T = \frac{2\pi}{\omega}$, $\theta = \theta_0$

Ex | Runner A takes T_A seconds to go around the circle.
" B takes $T_B > T_A$ seconds — — — —

How often does A overtake B? $\theta_{0,A} = \theta_{0,B} = 0$

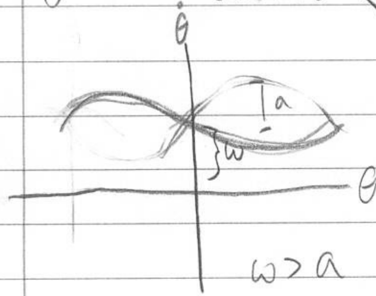
$\theta_A = \omega_a t + \theta_0 = \theta_B = \omega_b t + \theta_0 + 2\pi$

$\omega_a t = \omega_b t + 2\pi$; $\left(\frac{\omega_a - \omega_b}{2\pi}\right)^{-1} = t$ $\quad$ $T = \left(\frac{1}{T_a} - \frac{1}{T_b}\right)^{-1}$

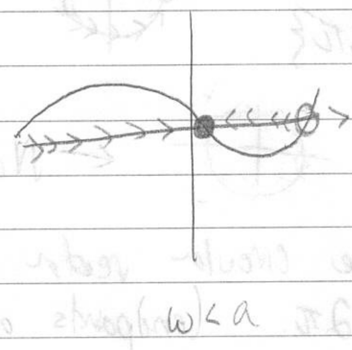
$\omega = 2\pi(\omega_a - \omega_b)$

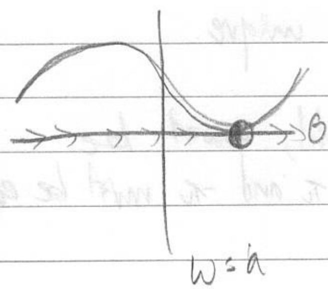
Non-Uniform Oscillation (Overdamped)

$\dot{\theta} = \omega - a\sin\theta$ — PLL's, apparently / Neural oscillation / Condensed Matt Physics

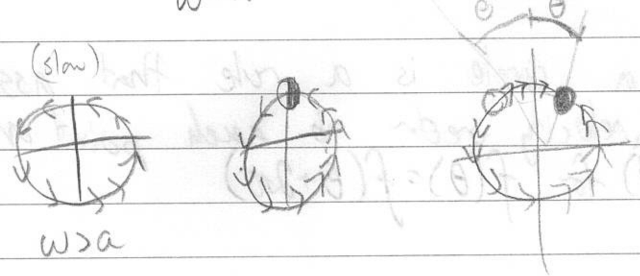


$\omega, a > \phi$





Saddle Node



§5 Intro To Linear Systems

2015/02/23

In 1D — flow on a line: $\dot{x}=f(x)$
— flow on a circle: $\dot{\theta}=f(\theta)$, $f(\theta)=f(2\pi+\theta)$

N-D systems are far richer in behaviour.

Simplest Case: $\left.\begin{aligned}\dot{x}&=ax+by\\ \dot{y}&=cx+dy\end{aligned}\right\}$ where a,b,c,d are parameters

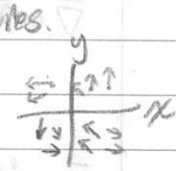
$$\boxed{\dot{\vec{x}}=A\vec{x} \quad (1)}$$

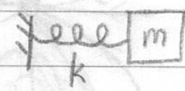
$\vec{x}=(x(t),y(t))$

$$\vec{x}=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix};\ \dot{\vec{x}}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=A\vec{x}$$

f.p. $x^*=\vec{0}$ always exists. Other fixed points are A's null space.

Phase Plane: Solutions of (1) are xy-plane trajectories.



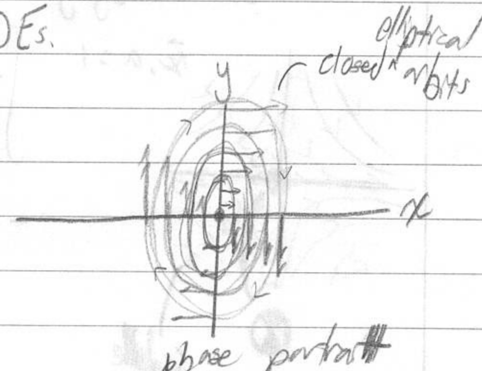
Ex | 

$m\ddot{x}+kx=0$

$\omega^2=\frac{k}{m}$

An n^{th}-order ODE $\Leftrightarrow$ n 1^{st}-order ODEs.

Set $\left.\begin{aligned}\dot{x}&=y\\ \dot{y}&=-\omega^2 x\end{aligned}\right\}\rightarrow \dot{\vec{x}}=\begin{bmatrix}0 & 1\\ -\omega^2 & 0\end{bmatrix}\vec{x}$



Let's extract behaviour without solving.

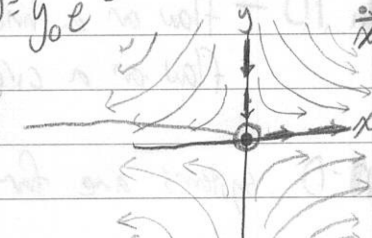
5.1.1 — $x^2+\omega^2 v^2=C\geq 0$

☆ Pseudo-spectrum analysis

Pseudo

Ex] $\dot{\vec{x}} = A\vec{x}$, $A = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}$ ⟵ uncoupled system

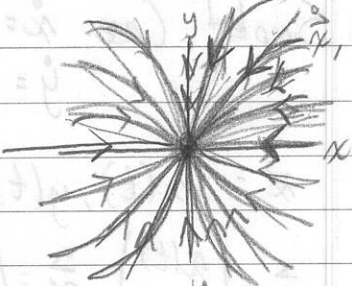
By standard integration, $x(t) = x_0 e^{at}$, $y(t) = y_0 e^{-t}$



∘ $\frac{dy}{dx} = \frac{\dot y}{\dot x} = -\frac{1}{a}\frac{y}{x}$

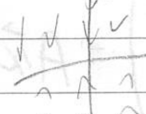
∘ Relative growth rates

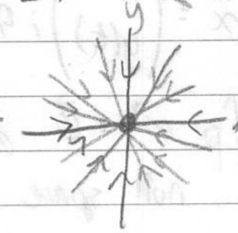
Any x_0, y_0 : $t \to \infty$, $\vec{x}^* = \vec{0}$



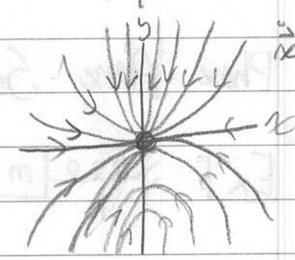
Divergence $= a - 1$

for $a < 1$, attracting





$\dot{\vec{x}} = \begin{cases} ax \\ -y \end{cases}$

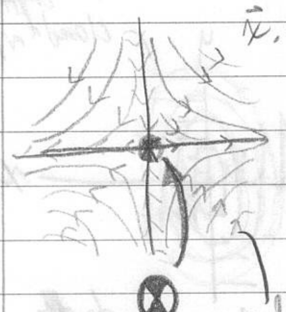


$\vec{\nabla} \circ \dot{\vec{x}} = \frac{\partial}{\partial x} ax + \frac{\partial}{\partial y} -y$

$= a - 1$

$\vec{\nabla} \circ \dot{\vec{x}} = \frac{\partial}{\partial x} \dot{x} + \frac{\partial}{\partial y} \dot{y}$

$\frac{dy}{dx} = -\frac{1}{a}\frac{y}{x}$



$\int -\frac{a}{y} dy = \int \frac{dx}{x}$

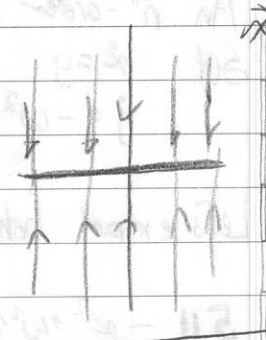
$-a\ln|y| = \ln|x| + C_1$

$\ln|y| = -\frac{1}{a}\ln|x| + C_2$

$y = e^{-\frac{1}{a}\ln|x| + C_2}$

$= C_3 x^{-1/a}$

$y = y_0 x^{-1/a}$



y-axis – stable manifold

x-axis – unstable

☆ $\vec{x}^* = \vec{0}$ stable node globally attractive all traj $\to \vec{0}$ on $t \to \infty$ for $r < 0$.

For $r \le 0$, Liapunov Stable:

All traj. that start near $\vec{x}^*$ remain close to $\vec{x}^*_0$ (no correlation)

§5 Eigenvalue Analysis

"It's your birthday and I give you a matrix." "This linear algebra is getting me [illegible] on right now."

Consider $\dot{\vec{x}} = A\vec{x}$ for $\vec{x} \in \mathbb{R}^2$.

- For diagonal matrices $A = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}$ solutions are decoupled and in the form $x(t) = c_1 e^{at} + c_2 e^{-t}$
- For general cases $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we wish to find a general solution of the form $e^{\lambda t}\vec{v} = \vec{x}$ — fixed parameters to be determined

If such solutions exist, then there would be exponential growth/decay depending on the sign of λ.

$\dot{\vec{x}} = A\vec{x}$, $\vec{x} = e^{\lambda t}\vec{v}$, $\dot{\vec{x}} = \lambda e^{\lambda t}\vec{v}$

$\lambda(\cancel{e^{\lambda t}}\vec{v}) = A(\cancel{e^{\lambda t}}\vec{v})$

$\boxed{\lambda\vec{v} = A\vec{v}}$ ⟵ Eigenvalues of A describe solutions of $\vec{x} = e^{\lambda t}\vec{v}$

$\vec{v} \in \operatorname{Ker}(A - I\lambda)$, $\det(A - \lambda I) = 0$

$\det(A - \lambda I) = \left|\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right| = (a-\lambda)(b-\lambda) - cb = 0$

$\lambda^2 - \underbrace{(a+d)}_{tr(A)}\lambda + \underbrace{(ad-bc)}_{\det(A)} = 0$ $\quad tr(A) = \sum \lambda_n$, $\det(A) = \prod \lambda_n$

For $A \in \mathbb{R}^{2\times 2}$, $\{\lambda\} = \dfrac{tr(A) \pm \sqrt{tr^2(A) - 4\det(A)}}{2}$

For a matrix $\in \mathbb{R}^{n\times n}$ with n distinct eigenvalues, M is a simple matrix if it has 2 eigenvectors.

If $\lambda_1 = \lambda_2 = \underbrace{\lambda_{m=1}}_{\text{algebraic multiplicity}}$, the geometric multiplicity is the number of distinct eigenvectors per value and is $\leq$ alg. mult.

Invertible matrices have two distinct eigenvectors.

$A = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda_1 = a$ $\vec{v} = \vec{e}_0$
$\lambda_2 = -1$, $\vec{v} = \vec{e}_1$

<u>Thm</u> If a matrix is diagonal, $\vec{e}_i$ are eigenvectors.

As long as $\vec{v}_1$ is lin. ind. to $\vec{v}_2$, the domain is spanned, and solutions can be represented in eigenspace as in
$\vec{x}_0 = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$ where $\alpha_1 = \vec{x}_0 \circ \vec{v}_1$; $\alpha_2 = \vec{x}_0 \circ \vec{v}_2$

$A(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = \alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2$

$\dot{\vec{x}} = A\vec{x}$. Try solutions in the form $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$:

$$\left.\begin{aligned} \dot{\vec{x}} &= c_1 \lambda_1 e^{\lambda_1 t} \vec{v}_1 + c_2 \lambda_2 e^{\lambda_2 t} \vec{v}_2 \\ Ax &= c_1 \lambda_1 e^{\lambda_1 t} \vec{v}_1 + c_2 \lambda_2 e^{\lambda_2 t} \vec{v}_2 \end{aligned}\right\} \dot{\vec{x}} = A\vec{x}$$

find $\vec{c}$ s.t. $\vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2$ $\quad \begin{bmatrix} | \\ \vec{x}_0 \\ | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\vec{c} = \begin{bmatrix} v \end{bmatrix}^{-1} \vec{x}_0$

<u>Ex</u> $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$
$\tau = -1$, $\Delta = -6$, $\lambda^2 + \lambda - 6 = 0$
$\lambda_{\{n\}} = \{2, -3\}$ $(\lambda - 2)(\lambda + 3) = 0$

$\vec{v}_1 : \lambda_1 = 2$
$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
$v_1 + v_2 = 2v_1$; $v_2 = 1$
$4v_1 - 2v_2 = 2v_2$; $v_1 = 1$
$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{v}_2 : \lambda_2 = -3$
$\vec{v}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

Fixed Points and Linearization

Recall:

$\dot{x} = f(x)$ has a fixed point x^*.

Let $\eta = x - x^*$, $\dot{\eta} = \dot{x} = f(x) \approx \cancel{f(x^*)}^{\,0} + f'(x^*)\eta$

$\dot{\eta} = f'(x^*)\eta$ determines stability.

For higher dimensions, only the Taylor expansion changes.

$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$ has a fixed point $\vec{x}^* = \begin{cases} x^* \\ y^* \end{cases}$

$u = x - x^*$; $\dot{u} = \dot{x} = f(x,y) = \cancel{f(x^*,y^*)}^{\,0} + \left.\frac{\partial f}{\partial x}\right|_{x^*,y^*}(x - x^*) + \left.\frac{\partial f}{\partial y}\right|_{x^*,y^*}(y - y^*) + O(\langle x,y\rangle^2)$

$v = y - y^*$; $\dot{v} = \dot{y} = g(x,y) = \cancel{g(x^*,y^*)}^{\,0} + \left.\frac{\partial g}{\partial x}\right|_{x^*,y^*}(x - x^*) + \left.\frac{\partial f}{\partial y}\right|_{x^*,y^*}(y - y^*) + O(\langle x,y\rangle^2)$

$\Rightarrow \dot{\vec{\eta}} = \left.\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\right|_{\vec{x}^*} \vec{\eta} \longrightarrow$ Jacobian Matrix

$\frac{dy}{dx} = \frac{-2y}{-x+x^3}$

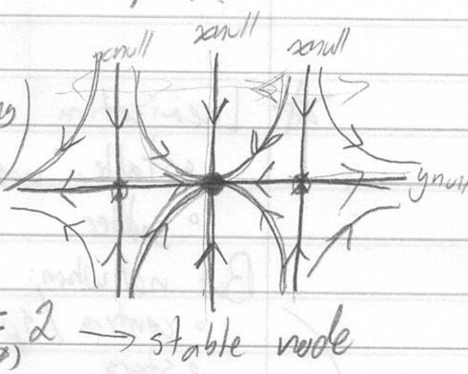
$\frac{dy/dx}{-x+x^3}$; $dx = \frac{1}{-2y}\,dy$

For a 2D flow $\dot{\vec{x}} = \vec{f}(\vec{x})$, the linearization at a fixed point $\vec{x}^*$ is

$\dot{\vec{\eta}} = A\vec{\eta}$ for $\vec{\eta} = \vec{x} - \vec{x}^*$; A is the Jacobian of $\vec{f}$.

Ex 1 $\vec{f}(\vec{x}) = \begin{cases} -x_1 + x_1^3 \\ -2x_2 \end{cases}$ ⇐ uncoupled

($\dot{y}$ is strong direction)



F.P.:

$x_2 = 0$, $x_1 = \{0, \pm 1\}$

$\vec{x}^* = \{(0,0); (1,0); (-1,0)\}$

$A = \begin{bmatrix} -1 + 3x_1^2 & 0 \\ 0 & -2 \end{bmatrix}$

$\tau_{(0,0)} = -3$, $\Delta_{(0,0)} = 2 \longrightarrow$ stable node

$\tau_{(\pm1,0)} = 0$, $\Delta_{(\pm1,0)} = -4 \longrightarrow$ saddle point

$A\big|_{x^*=(0,0)} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$; $A\big|_{(\pm1,0)} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

However, does this work?

$x_2(t) = ce^{-2t}$, $\lim_{t\to\infty} x_2(t) = 0$

We can now solve $x_1(t)$ as a seperate system.

?→ RK: for non-border cases in the Δ-τ diagram, the nonlinear sys behaves the same as the linearized system *! near the fixed points!

<u>Ex</u> $\dot{x} = -y + ax(x^2+y^2)$ $\quad J = \left[\text{something messy}\right]$

$\dot{y} = x + ay(x^2+y^2)$

<u>FP</u>

$\vec{x}^* = (0,0)$ $\quad$ for $x^* = (0,0)$, $\vec{\eta} = \vec{x}$. This way, we can discard higher-order terms of $\vec{f}$.

$$\vec{f}\Big|_{x^*=(0,0)} = \begin{cases} -y \\ x \end{cases} ; \quad A\Big|_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This is a spiral, a border case, so the linearization is invalid. We switch to polar coords:

$$\left.\begin{aligned} \tfrac{d}{dt}\, r\cos\theta &= -r\sin\theta + ar^3\cos\theta \\ \tfrac{d}{dt}\, r\sin\theta &= r\cos\theta + ar^3\sin\theta \end{aligned}\right\} \text{chain rule, gets you}$$

$\dot{r} = ar^3$, $\dot{\theta} = 1$. for $a = 0$, center. $a < 0$, stable spiral, $a > 0$ unstable spiral

☆ Linearization works when:

- stable, unstable nodes $(\Delta > 0)$
- saddles

$\Big\}$ $Re(\lambda) \neq 0$ $\boxed{\text{Hyperbolic}}$ ("nice")

But not when:

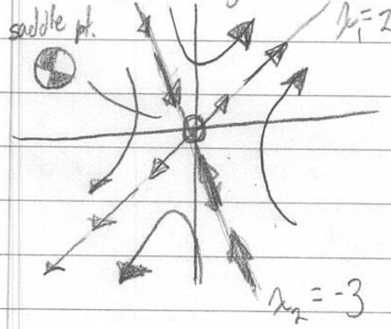
- centers / spirals
- stars
- Degenerate nodes → only one eigenvector

$\Big\}$ $Re(\lambda) = 0$ $\boxed{\text{Marginal / Degenerate}}$

→ must look at higher orders $(x_1 - x_1^*)^2$, $(x_2 - x_2^*)^2$; $(x_1 - x_1^*)(x_2 - x_2^*)$

Ex | Draw the phase portrait for $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$.

From previously: $\left\{ 2 : \begin{bmatrix} 1 \\ 1 \end{bmatrix}, -3 : \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\}$, $\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$



for $(0, y)$:

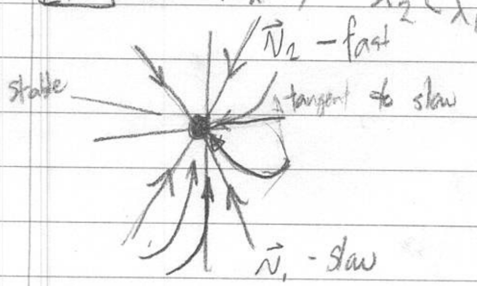
$x(t) = c_1 (e^{2t} - e^{-3t})$

$y(t) = c_1 (\underbrace{e^{2t}}_{\text{diverge}} + \underbrace{4e^{-3t}}_{\text{converge}})$

$\lim_{t \to \infty}$ $\vec{N}_1$ diverges, $\vec{N}_2$ converges

$\operatorname{div} \dot{\vec{x}} = \operatorname{tr}(A)$

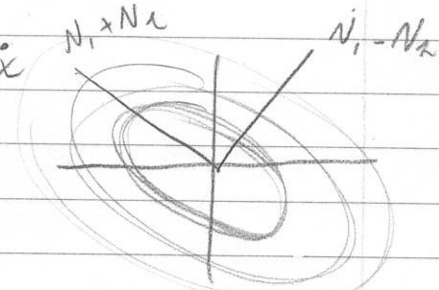
Ex | $\dot{\vec{x}} = A\vec{x}$; $\overbrace{\lambda_2}^{\text{fast}} < \overbrace{\lambda_1}^{\text{slow}} < 0$ $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$



Trajectory approaches $\vec{0}$ tangent to the slower (smallest absolute value) eigendirection.

Ex | $\ddot{x} + \omega^2 x = 0$; $A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$ for $v = \dot{x}$

$\tau = 0$, $\Delta = \omega^2$, $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$



$x(t) = c_1 e^{i\omega t} \vec{N}_1 + c_2 e^{-i\omega t} \vec{N}_2$

$c_1 \vec{N}_1 \cos \omega t + i c_1 \vec{N}_1 \sin \omega t + c_2 \vec{N}_2 \cos \omega t - i c_2 \vec{N}_2 \sin \omega t$

$\underbrace{c_1 \vec{N}_1 + c_2 \vec{N}_2}_{\text{"x"}}$ $\underbrace{(c_2 \vec{N}_2 - c_1 \vec{N}_1}_{\text{"y"}}$

$\dot{\vec{x}} = A\vec{x}$, $\vec{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\lambda_{1,2} = \frac{1}{2}\tau \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta}$ where $\tau = Tr(A) = \lambda_1 \lambda_2$, $\Delta = det(A) = \lambda_1 \lambda_2$

$\vec{v}_1 \sim \lambda_1$; $\vec{v}_2 \sim \lambda_2$: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$; c_1, c_2 from I.C.

For $4\Delta > \tau$, $\lambda_{1,2} = \alpha + \omega j$, $\alpha = \frac{1}{2}\tau$, $\omega = \sqrt{4\Delta - \tau^2}$

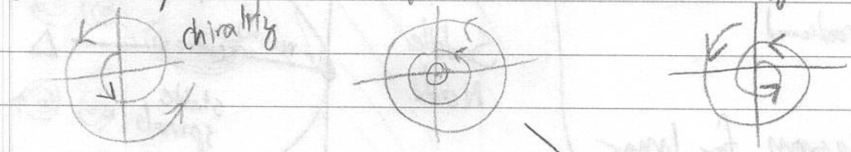
When $\alpha = 0$, $\vec{x}$ forms ellipses (phasors)

Algebraic multiplicity: just touches axis, degenerate solutions.

$\alpha = Re(\lambda) > 0$: outwards spirals

$\alpha = Re(\lambda) = 0$: ellipses/centres

$\alpha = Re(\lambda) < 0$: inwards spirals



(check flow at a point for direction) $T = \frac{2\pi}{\omega}$

For $4\Delta = \tau$; $\lambda_1 = \lambda_2 = \frac{1}{2}\tau$:

If the alg. mult is 1, geo. mult could be one or two.

One eigenvector: Jordan Form ("defective matrices")

Ex) $A = \lambda I$; $\vec{v} = \vec{e}$. If $A = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix}$; $\lambda_1 = \lambda_2 = \lambda$. $x \, i \rho \alpha \lambda$

What does the phase portrait look like?

$\boxed{x(t) = e^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_2}$

A) Case $\vec{v}_1 \neq \vec{v}_2$, two geo. mult:

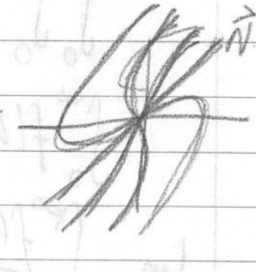
$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$

$A\vec{x}_0 = \lambda\vec{x}_0$ so any x_0 is an eigenvector, $A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Form is a star (either converging or diverging)

B) Case $\vec{v}_1 = \vec{v}_2 = \vec{v}$, one geo. mult:

A is $\begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix}$ for $b \neq 0$. degenerate node



If $\Delta < 0$:

Two real, opposite signs (Saddle Nodes)

If $\Delta > 0$:

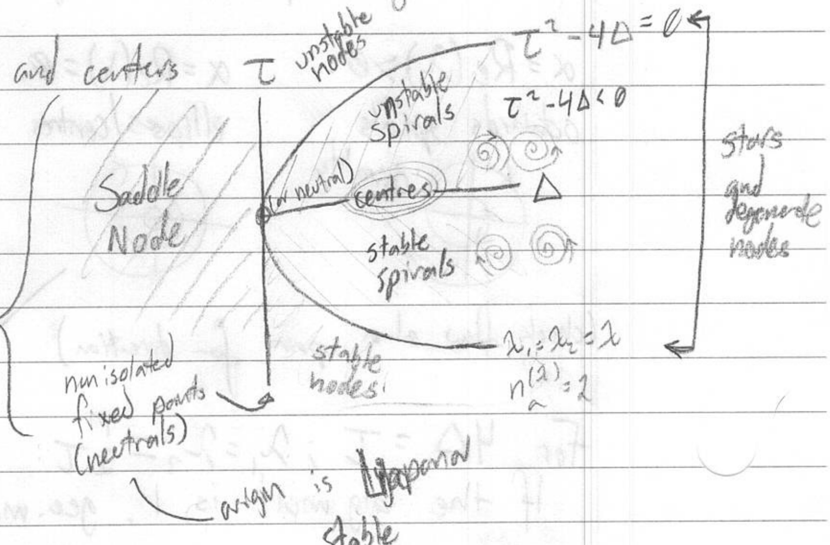
- Two Real
  - Positive — Diverging
  - Negative — Converging

  } conics and stars and degenerate nodes

- One Imaginary — Spirals and centers

$\tau^2 - 4A$ } real/cmnd

Stability Diagram for linear systems in 2 variables



Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Classify dynamics.

$\rightarrow \tau = 5,\ \Delta = -2$

Since $\Delta < 0$, $\vec{x}$ is a saddle node.

Ex $A = \begin{bmatrix} 7 & 3 \\ 1 & 5 \end{bmatrix}$

$\rightarrow \tau = 12,\ \Delta = 32 \Rightarrow \lambda = \{8, 4\} \Rightarrow$ Unstable node.

Ex $A = \begin{bmatrix} 10 & 1 \\ 0 & 9 \end{bmatrix}$ (upper-triangular)

$\tau = 19,\ \Delta = 90 \Rightarrow \lambda = \{9, 10\} \Rightarrow$ Unstable node

$$\int_0^\infty \int_0^{f(\Delta)} 1 \, d\tau \, d\Delta \qquad \int_0^\infty \int_{f(\Delta)}^\infty 1 \, d\tau \, d\Delta$$

$$= \int_0^\infty f(\Delta) \, d\Delta \qquad \lim_{x \to \infty} \int_0^\infty x - f(\Delta) \, d\Delta$$

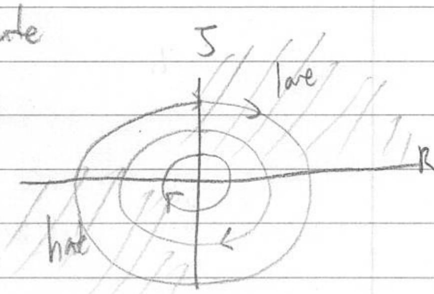
$$\lim_{x \to \infty} \int_0^\infty \frac{f(\Delta)}{x - f(\Delta)} \, d\Delta$$

§5.3 Love Affairs

R loves J, but J is fickle.

$R(t)$: love of R for J $R, J > 0$ love
$R < 0$ Hate
$J < 0$ Hate

$\dot{R} = \alpha J$
$\dot{J} = -\beta R$ for $\alpha, \beta > 0$



$\vec{x} = \begin{pmatrix} R \\ J \end{pmatrix}$, $\dot{\vec{x}} = \begin{bmatrix} 0 & \alpha \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$, $\lambda = 0 \pm i\sqrt{\alpha\beta}$

$\boxed{\text{Ex}}$ $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, symmetric perception: α - own feelings (cautiousness)
β - partner's feelings
$a < 0$, $b > 0$

$\tau = 2a$, $\Delta = a^2 - b^2$, $\tau^2 - 4\Delta = 4b^2 > 0$
< 0 $\lambda = \{(a+b), (a-b)\}$

Depending on Δ, either saddle node or stable nodes
for $|b| > |a|$ for $|b| < |a|$

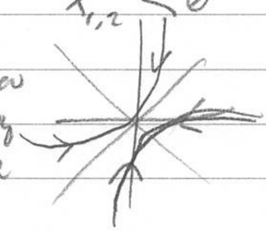
$|b| > |a|$: ∞ love

∞ hate

~~mutual~~ "relationship either ends really great or really terrible if you care a lot and are not cautious"

"if you are too cautious, you have total apathy no matter where you start"

$\lambda_{1,2} < 0$



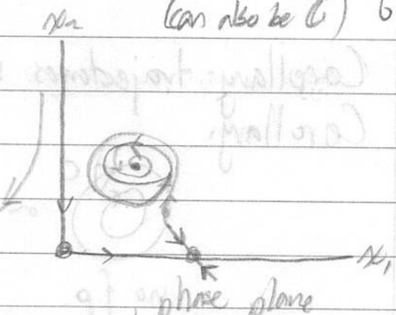
§6.1 2D Non-Linear Flow 03-04-2015

Guest Lecturer:
Ting Zhou
t.zhou@ned.edu

Recall for 1-D flows, $\dot{x} = f(x)$ ← time invariant
- Existance and uniqueness of solutions
- f.p. and linear stability analysis
- Bifurcation theory

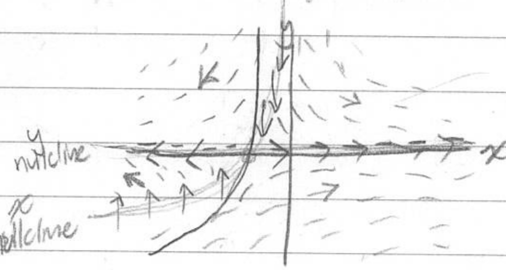
General form: $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x} \in \mathbb{R}^n$, $\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$
for linear systems, $\vec{f}(\vec{x}) = A\vec{x}$ (can also be $\mathbb{C}$)

$.91\bar{6}$
$6 \overline{)5.5}$
54
10
6
40
36



Interested in:
fixed points, stable orbits, convergence and stability

Ex $\dot{x} = x + e^{-y}$
$\dot{y} = -y$



Fixed points:
$-y = 0 \to y = 0$
$x + 1 = 0 \to x = -1$
$(-1, 0)$

$y(t) = ce^{-t}$, $\lim_{t\to\infty} y(t) = 0$
$\lim_{t\to\infty} \dot{x} = x + 1$
for long periods, and on the x-axis, $\dot{x} = x+1$, $x = -1$ is unstable.

Nullclines:
$\dot{x} = 0$ or $\dot{y} = 0$.

$\dot{x} = 0$:
$x = -e^{-y}$, $y = -\ln(-x)$

$\dot{y} = 0$:
$y = 0$.

Existance and Uniqueness (n-dim flow, $n \geq 1$)

Thm

$$\text{IVP}\begin{cases}\dot{\vec{x}} = \vec{f}(\vec{x}) \\ \vec{x}(0) = \vec{x}_0\end{cases}$$

Suppose $\vec{f}$ is differentiable for $\vec{f} \in C_1$, $\frac{\partial f_i}{\partial x_j}$ is cont. for $i,j = 1..n$

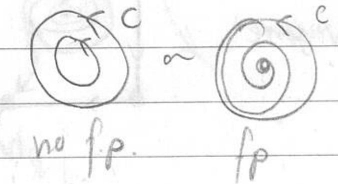
$\vec{x} \in D \subseteq \mathbb{R}^n$. Then IVP has a

D: ← open bounded

unique solution $\vec{x}(t)$ if $\vec{x}_0 \in D$

Corollary: trajectories never cross.

Corollary:



Cannot leave orbit (can't cross boundary)

§6.4 Sheep vs Rabbits

No predation, only resource competition.

$R=0: \dot{S} = k_S S(1-S) \rightarrow$ Logistic Model

$S=0: \dot{R} = k_R R(1-R) \rightarrow$ Logistic Model $(K_R >> K_S)$

Let $\vec{x}(t) = \begin{bmatrix} R \\ S \end{bmatrix}$; $\dot{\vec{x}} = \begin{pmatrix} x_1(\underbrace{3-x_1}_{\text{logistic}} \underbrace{-2x_2}_{\text{interaction}}) \\ x_2(2-x_2-x_1) \end{pmatrix}$

Fixed points:

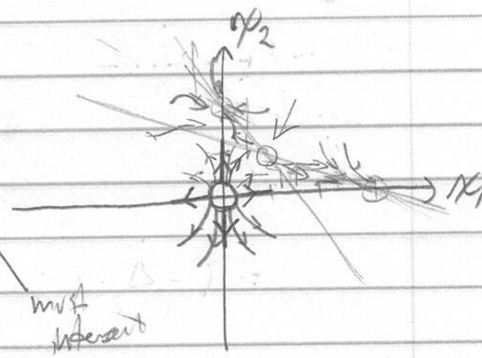
$\vec{x}^* = (0,0)$

$= \cancel{(3,0)}$ $\quad x_1 + 2x_2 = 3$

$= \cancel{(0,2)}$ $\quad x_1 + x_2 = 2$ — must intersect

$= \cancel{(3-2x_2, x_2)}$ $\quad (3,0)$

$= \cancel{(2-x_2, x_2)}$ $\quad (0,2)$, $(1,1)$

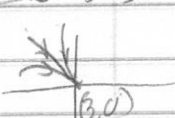


$$A = \begin{bmatrix} 3-2x-2y & -2x \\ -y & -x-2(y-1) \end{bmatrix}$$

$\vec{x}^* = (0,0)$: $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $\lambda = \{3, 2\}$ → unstable (eigenvectors $(1,0)$, $(0,1)$)

~~$\vec{x}^* = (x_1, 2-x_1)$: $A = \begin{bmatrix} 3-2x-4+2x & -2x \\ x-2 & -x-2x \end{bmatrix} = \begin{bmatrix} -1 & -2x \\ x-2 & -3x \end{bmatrix}$~~

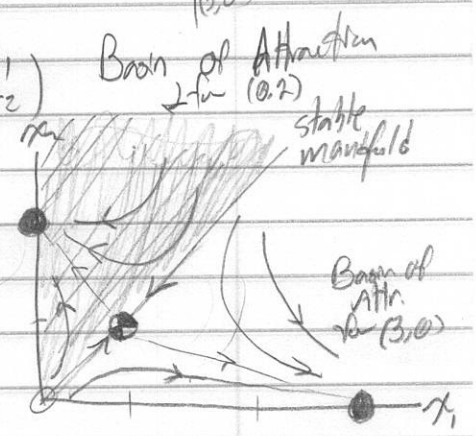
$\vec{x}^* = (3,0) \rightarrow A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \rightarrow \lambda = -3$, $\lambda = -1$, $\vec{v}_{-1} = (3,-1)$



$\vec{x}^* = (0,2) \rightarrow A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \rightarrow \lambda_1 = -1$, $\lambda_2 = -2$, $\vec{v}_{-2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\vec{x}^* = (1,1) \rightarrow A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$ $\lambda = -1 \pm \sqrt{2}$

$\Delta = -1$, $\tau = -2$ } saddle



Principle of Competitive Exclusion

stable manifold that defines saddle-node: separatrices

Conclusions:

- One species gets extinct w/ high probability while the other reaches carrying capacity.

Ignores:
seasons, external influences, lifecycles

☆ Basin of Attraction of a fixed point x^* is the set of initial conditions $\{x_0\}$ s.t. $\vec{x}(t) \xrightarrow[t\to\infty]{} \vec{x}^*$

§6.5 Conservative Systems

Conservative fields: ◦ $F(x) = -\vec{\nabla} V(x)$
◦ Independent of Path
◦ Curl is 0.

In 1-D, $F(x) = m\frac{d^2x}{dt^2} = -\frac{d}{dx}V(x)$

$\boxed{\text{Lenz Vector}}$

$m\ddot{x} + \frac{d}{dx}V(x) = 0$

$m\ddot{x}\dot{x} + \dot{x}\frac{dV}{dt} = \frac{1}{2}m(\dot{x})^2 + \dot{x}\frac{dV}{dx}$

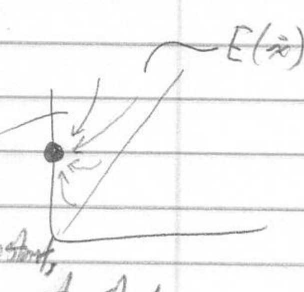
$= \underbrace{\frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V(x)\right]}_{\text{Total Mech energy}} = \frac{dE(x)}{dt} = 0$

Since $\frac{dE(x)}{dt} = 0$, Energy is conserved. Note that this is time-invariant!

$E(x)$ is constant and depends on initial conditions.
The conserved quantity is a real-valued function $\vec{E}(\vec{x})$ st $\dot{E} = 0$. & $E(\vec{x})$ is non-const in an open neighborhood.

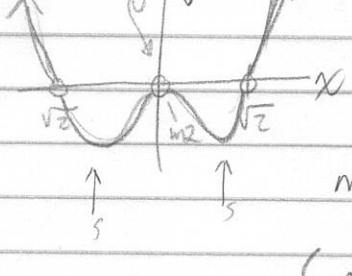
Eg.1 A conservative system cannot have an attractor.

Pf: Suppose a fixed point $\vec{x}^*$ is attractive.



if this is true, $E(\vec{x})$ must be a constant, since its value at start must be the same as the end.

Ex 2 $m=1$, $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$

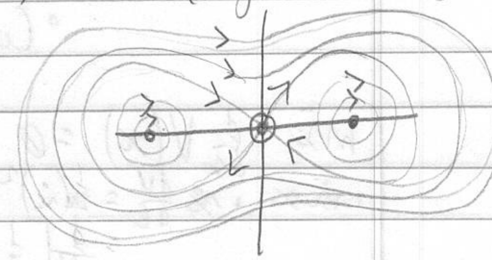


$m\ddot{x} = f(x) = x - x^3$

$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases}$ $\vec{x}^* = (0,0), (\pm 1, 0)$

$A = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix} \Rightarrow A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \lambda_{1,2} = \pm i, \Delta = -1$ (saddle node)

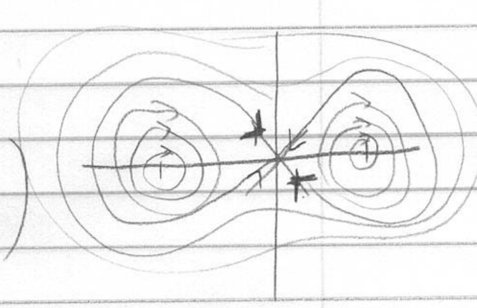
$A_{(\pm 1, 0)} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} : \Delta = 2, T = 0$ ~~(degenerate case)~~ ~~candles~~ centres



Ex 1 $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$, $V_x(x) = -x + x^3$

$\ddot{x} - x + x^3 = 0$

$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases} \rightarrow \vec{x}_1^* = (0,0)$, $\vec{x}_2^* = (\pm 1, 0)$



$$A = \begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial(x-x^3)}{\partial x} & \frac{\partial(x-x^3)}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$$

$$A\big|_{\vec{x}_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \quad A\big|_{\vec{x}_2^*} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

$\lambda = \pm 1$ $\qquad$ $\lambda = \pm 2i$

$\lambda = 1: \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ $\qquad$ $\lambda = -1: \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ $\qquad$ $\left.\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right.$

$x - y = 0$ $\qquad$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ $x + y = 0$ $\qquad$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad$ $x = 0$ y can vary

$x = y$

$0 \quad 0$

~~$x + y = 0$~~

$x - y = 0$

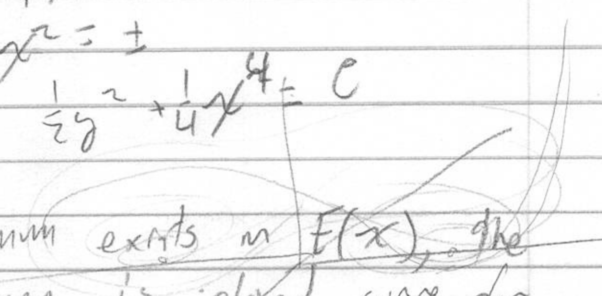
$E(x) = \frac{1}{2}\dot{x}^2 + V(x) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$

for small x, $\frac{1}{2}y^2 = \frac{1}{2}x^2$, lines of slope ± 1

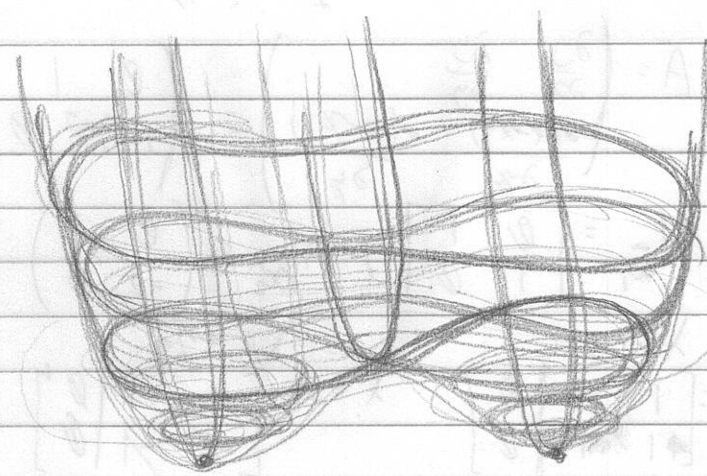
for large x, $-\frac{1}{2}y^2 = -\frac{1}{4}x^4$

$\frac{1}{2}x^2 = \pm$

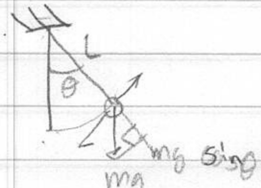
ovals $\qquad \frac{1}{2}y^2 + \frac{1}{4}x^4 = C$



Theorem: If a local minimum exists in $E(x)$, the flow around the minimum is closed curve for conservative functions.



§ 6.7



$h = L - \cos\theta$

$$\frac{\partial^2 \theta}{\partial t^2} + \frac{g}{L}\sin\theta = 0$$

$$\omega = \sqrt{g/L}, \quad \tau = \omega t, \quad d\tau = \omega dt$$

potential = $V = -\cos\theta$

$$\frac{d^2\theta}{dt^2} = \omega^2 \frac{d^2\theta}{d\tau^2} \Rightarrow \cancel{\omega^2}\frac{d^2\theta}{d\tau^2} + \cancel{\omega^2}\sin\theta = 0$$

$$\frac{d^2\theta}{d\tau^2} + \sin\theta = 0 \longrightarrow \begin{cases} \dot\theta = v \\ \dot v = -\sin\theta \end{cases}$$

$\ddot\theta = -\sin\theta \longrightarrow$ $\frac{d}{d\tau}V(\theta) = \sin\theta$, so $V(\theta) = -\cos\theta$

f.p.: $(\theta^*, v^*) = (k\pi, 0)$ for $k \in \mathbb{Z}$

$\hookrightarrow \{(0,0), (\pi, 0)\}$

$$A = \begin{bmatrix} 0 & 1 \\ -\cos\theta & 0 \end{bmatrix}; \quad A\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \quad A\Big|_{(\pi,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$(\theta^*, v^*) = (0,0)$: $\lambda^2 + 1 = 0$; $\lambda = \{\pm i\} \rightarrow$ Centers

$= (\pi, 0)$: $\lambda = \{\pm 1\} \rightarrow$ Saddles.

$$\dot\theta \frac{d^2\theta}{d\tau^2} + \dot\theta \sin\theta = 0$$

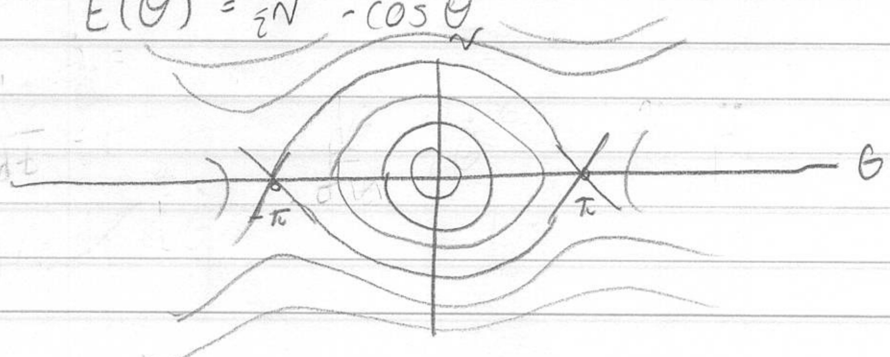
$$\frac{d\theta}{d\tau}\frac{d^2\theta}{dt^2} + \frac{d\theta}{d\tau}\sin\theta$$

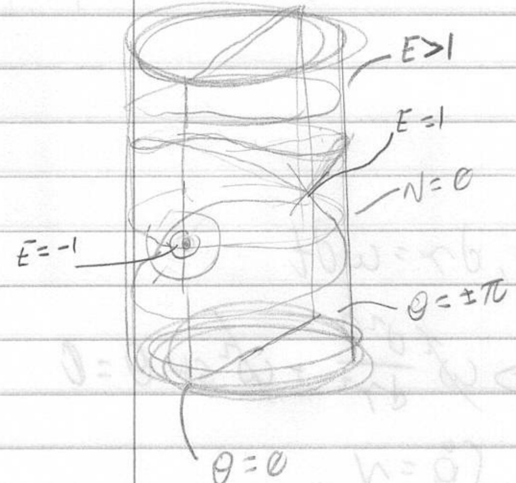
$$\frac{d}{dt}\left(\frac{d^3\theta}{dt^2} + d\theta\sin\theta\right)$$

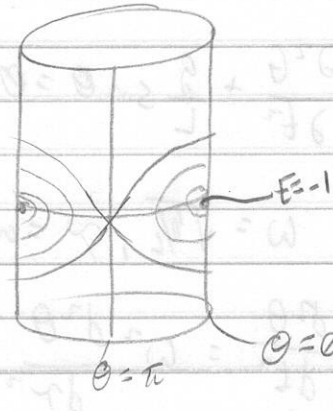
$$\frac{d}{d\tau}\underbrace{\left(\tfrac{1}{2}\dot\theta^2 - \cos\theta\right)}_{E(\theta) = \frac{1}{2}v^2 - \cos\theta} = 0$$

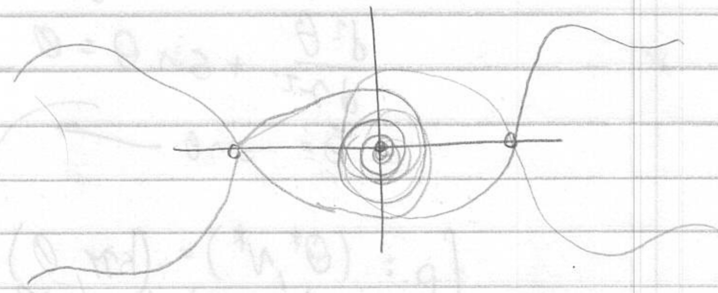
Also $E(\ldots)$: $\underbrace{\tfrac{1}{2}\dot\theta^2 - \cos\theta}_{E(\theta, v)}$

(less constants)









$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$$
$b > 0$

$$\frac{d}{d\theta} V = \ddot{\theta} = -\sin\theta - b\theta$$
$$V = -\cos\theta - b\theta$$

$$\frac{1}{2}\dot{\theta}^2 + \cos\theta - b\dot{\theta}$$
$$\frac{1}{2}N^2 - \cos\theta - b\theta$$

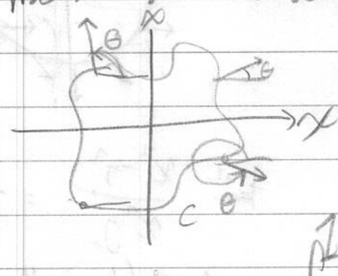
§6.8 Index Theory

Buy Geometry of Music

$\boxed{\text{Quiz: April 6th}}$

$\boxed{\text{Bruce Kusse, mathematical Physics}}$

Index theory gives a <u>global picture of the stability</u>.
The <u>index</u> of a closed curve C is an integer that measures the winding of the vector field.



The first and last angle must match, so we can only move an integer number of turns from start to end.

$$\int_0^1 \theta'(\vec{x}(t))\,dt \quad \text{for } t = [0,1];\ \vec{x}(0) = \vec{x}(1)$$

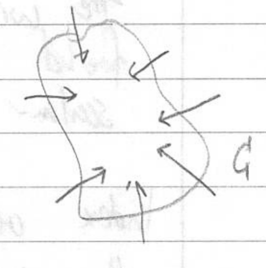
$$I_C = \frac{[\theta]_C}{2\pi} \quad \text{— net change in } \phi \text{ over the circuit}$$

$$\begin{array}{cc} 8 & 10 \\ 4 & 4 \\ \hline \boxed{32} & 40 \end{array}$$

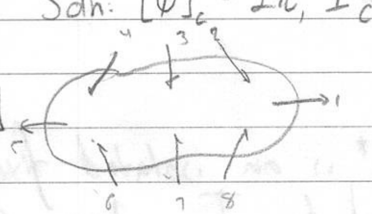
<u>Exam</u> Let $\dot{\vec{x}} = f(\vec{x})$; choose arbitrary C (simple, closed curve). At each point on C, the vector field $\dot{\vec{x}} = (\dot{x}, \dot{y})$ is the vector field, and the angle $\phi = \tan^{-1}(\dot{y}/\dot{x})$ is measured from the horizontal axis. $(\phi \in [0, 2\pi))$

As $\vec{x}$ (any point) moves clockwise (CCW) on C, ~~and~~ ϕ changes continuously, since the vector field is smooth.

Ex] Sink: $I_C = ?$
Soln: $[\phi]_C = 2\pi$, $I_C = +1$

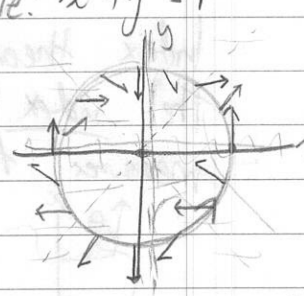


Ex]



$\Rightarrow I_C = -1$

Ex] $\dot{x} = x^2 y$, $\dot{y} = x^2 - y^2$. C is the unit circle: $x^2 + y^2 = 1$



$I_c = 0$

(note that this includes a fixed point.)

Properties:

① Suppose C is ~~cont diff~~ continuously deformed into C'; then $I_c = I_{c'}$

$\tan^{-1} \frac{x^2 - y^2}{x^2 y} = \tan^{-1} \frac{1 - \frac{y^2}{x^2}}{y}$ $2\sqrt{2}$,

$= \tan^{-1}\left(\frac{1}{y} - \frac{y}{x^2}\right)$

(i) If angles change smoothly, then $I = \frac{[\phi]_{c'}}{2\pi}$ should change slowly. But I_c is an integer, it cannot change w/o jumping.

$\frac{1}{1 + x^2}$

② If C does not enclose a fixed point, $I_c = 0$.

$\frac{1}{1 + (y^{-1} - yx^{-2})^2}$

$1 + y^{-2} - 2x^{-2} + y^2 x^{-4}$

③ If we reverse the vector field, the index is the same.

$1 + \sin^{-2}\theta - 2\cos^{-2}\theta$

④ Suppose C is a closed orbit, ccw trajectory, such that the tangent to the curve is equal to the vector field at all points. (or a ~~true~~ scalar multiple), $I_c = +1$

$1 + \sin^2\theta + 2\cos^2\theta$

$\sin^2\theta + \cos^2\theta + 2\cos^2\theta + \sin^2\theta$

$2\sin^2\theta + 2\cos^2\theta + \cos^2\theta$

$2 + \cos^2\theta$

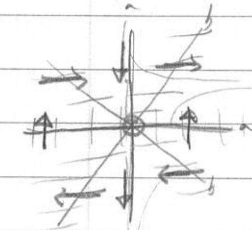
Index of a point: suppose $\vec{x}^*$ is an isolated fixed point, then the index I of x^* is defined by I_c where C is any closed curve around x^*, and no other fixed points.

Ex) find the index of a stable node, unstable node, and a saddle.

As well as spiral, degenerate, etc

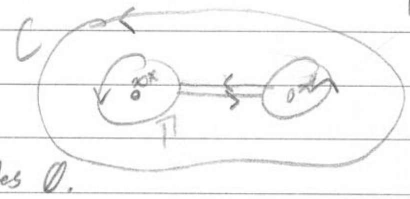
$\underbrace{\text{Stable: } +1 \quad \text{Unstable: } +1}_{\text{cannot infer stability}}$ Saddle: -1

If a closed curve C surrounds n f.p.s $\{\vec{x}_i^*\}_{i=1}^{n}$, then $I_c = \sum_{k=1}^{n} I_k$

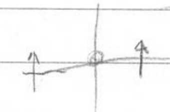


If C encloses $\{\vec{x}^*_{1..n}\}$, then $I_C = \sum_{k=1}^{n} I_k$

$I_{\Gamma}^{\boxed{1}} = I_C$



Bridge contributes 0.



$x^2 y = C \quad x^2 - y^2 = C$

$x^2 - y^2 = x^2 y$

$\cancel{x^2 = x^2 y}$

$(x^2 + 2xy + y^2) = x^2 y + 2xy + 2y^2$

$(x + y)$

$(x+y)(x-y) =$

$(x+y)^2 = y(x^2 + 2x + 2y)$

$= y(x^2 + 2x + 1 + 2y - 1)$

$(x+y)^2 = y([x+1]^2 + 2y - 1)$

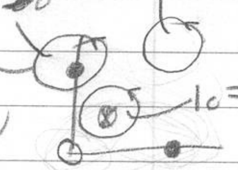
Thm Any closed orbit in the phase plane must enclose f.p. whose indices sum to $+1$.

Corollary:

- $\exists$ at least one f.p. inside a closed orbit
- If $\exists$ one f.p. inside the closed orbit, it cannot be just a saddle.
- We can rule out the possibilities of closed orbits.

Ex) $\dot{x} = x(3 - x - 2y)$, $\dot{y} = y(2 - x - y)$; $x, y \geq 0$

$\vec{x}^* = \{\underset{\text{unstable}}{(0,0)}, \underbrace{(0,2), (3,0)}_{\text{stable}}, \underset{\text{saddle}}{(1,1)}\}$



$I_c = +1$ cannot be closed

$I_c \neq +1$

$I_c = -1$

No closed orbits.

§7 Limit Cycles

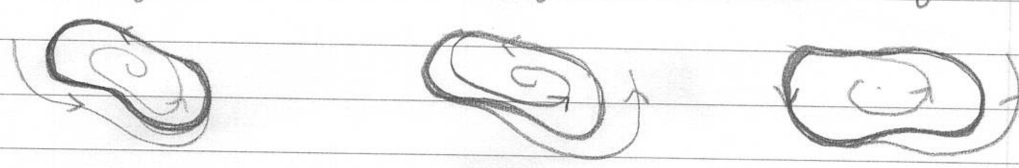
Mar 25^{th}, 2015

<u>Def</u>: A limit cycle is an isolated closed orbit / trajectory
(An isolated closed trajectory means $\nexists$ closed trajectories in the neighbourhood. Surrounding trajectories spiral into or away from the trajectory.

Attractor (stable) Limit Cycle:

Repulsive (unstable) Limit Cycle:

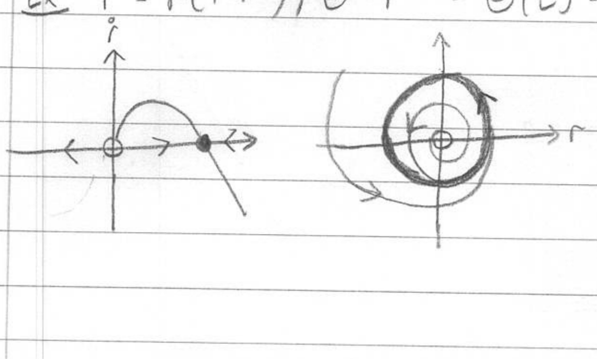
Semi-Stable Limit Cycle:



<u>Applications</u>: Systems w/ self-sustained oscillations have stable L.C.; i.e., attractors.
<u>Ex:</u> Heartbeat, oscillating chemical reactions, LC oscillators, human body temperature
$\exists$ a preferred frequency and amplitude that the system falls to.

In centres, the closed trajectories are determined by initial conditions. With attractors, the trajectory depends on the <u>structure</u> & <u>function</u> of the system. Note that attractors cannot occur with linear systems as they do not possess isolated trajectories.

<u>Ex</u> $\dot{r} = r(1-r^2)$; $\dot{\theta} = 1 \rightarrow \theta(t) = t + \theta_0$ ($r \geq 0$)



$0 < r < 1 \Rightarrow r \to 1, t \to \infty$
$r > 1 \Rightarrow r \to 1, t \to \infty$

Ex 1 Van der Pol Eqn: $\ddot{x} + \underbrace{\mu(x^2-1)\dot{x}}_{\text{nonlinear deviation from SHO}} + x = 0, \; \mu \geq 0$

$\mu = 0 \rightarrow$ SHO

$\mu > 0 \rightarrow |x| > 1 \rightarrow$ Ordinary Damping

$\qquad\quad\; |x| < 1 \rightarrow$ Growing Solutions

§7.2 Ruling Out Closed Orbits

Three methods to rule out closed trajectories:

1. Gradient Systems: $\dot{\vec{x}} = -\nabla V(\vec{x})$; $V: \mathbb{R}^n \to \mathbb{R}$, differentiable

<u>Thm</u>: Closed orbits are impossible in gradient systems.

<u>Proof</u>: $\Delta V = 0$ after one period.

$$\Delta V = \int_0^T \dot{V}\,dt = \int (\nabla V \cdot \dot{\vec{x}})\,dt = -\int |\dot{\vec{x}}|^2\,dt \leq 0$$

$\because \Delta V \neq 0$ for $\dot{\vec{x}} \neq \vec{0}$, no closed orbits exist.

<u>Ex</u>: Show $\nexists$ c. orbits for $\dot{x} = \sin y$, $\dot{y} = x\cos y$

$V(x,y) = -x\sin y \Rightarrow -\vec{\nabla} V = \vec{f}(\vec{x})$

$\dot{\vec{x}}$ is a gradient system, so no closed orbits exist.

2. Lyapunov Functions: $\dot{\vec{x}} = \vec{f}(\vec{x})$, f.p. $\vec{x}^*$

<u>Def</u>: If $\exists$ a function $V(\vec{x}): \mathbb{R}^n \to \mathbb{R}$ that satisfies the below:

1. $V(\vec{x}) > 0 \; \forall \vec{x} \neq \vec{x}^*$, $V(\vec{x}^*) = 0$
2. $\dot{V}(\vec{x}) < 0 \; \forall \vec{x} \neq \vec{x}^*$

(However, there is no surefire way to find $V(\vec{x})$)

Then $\vec{x}^*$ is a global stable f.p.

<u>Ex</u>: Construct a Lyap. function to show that $\nexists$ closed orbit on $\dot{x} = -x + 4y$, $\dot{y} = -x - y^3$

Suppose $V(x,y) = x^2 + ay^2$.

$$\dot{V} = 2x\dot{x} + 2ay\dot{y} = 2x(-x+4y) + 2ay(-x-y^3)$$

$$= -2x^2 + 2xy(4-a) - 2ay^4 \quad \to \text{Let } a = 4 \text{ to force } \dot{V} < 0$$

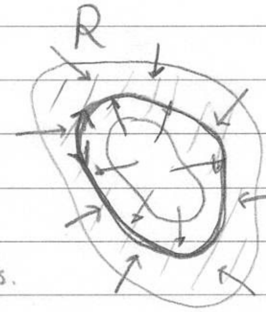
$\dot{V}(x,y) = -2x^2 - 8y^4 < 0$, $V(x,y) > 0$

3. Dulac's Criteria: $\dot{\vec{x}} = \vec{f}(\vec{x})$.

<u>Def</u> If $\exists\, g(\vec{x})$ s.t. $\vec{\nabla} \cdot (g\dot{\vec{x}})$ has the sign over any region R, then $\nexists$ a closed orbit on R.

§7.3 Poincaré Bendixson Theorem

Show that there exists a limit cycle.



trivial
(1) R is a closed, bounded region or subset of the plane.
(2) $\dot{\vec{x}} = \vec{f}(\vec{x})$ is a cont. diff vector field on R.
(3) R does not contain any fixed points.

tricky → (4) $\exists$ a C trajectory living entirely inside R $\forall t$

Then either C is a closed orbit or it will spiral toward a closed orbit.

(See Jordan and Smith 19{78/87} for topoproof)

Ex $\dot{r} = r(1-r^2) + \mu r\cos\theta$, $\dot{\theta} = 1$, $\mu > 0$. Show $\exists$ a limit cycle.

Let $R = \{r, \theta \mid 0 < r_{min} \leq r < r_{max}\}$

$\dot{r} = r_{min}(1-r_{min}^2) + \mu r_{min}\cos\theta > 0$

$= \cancel{r_{min}}(1-r_{min}^2) - \mu \cancel{r_{min}} > 0$

$r_{min}^2 < \mu$; $r_{min} < \sqrt{1-\mu}$

Similarly,

$r_{max} < \sqrt{\mu+1}$,

$\mu \leq 1$.

Pick something like $r_{min} = 0.999\sqrt{1-\mu}$, $r_{max} = 1.001\sqrt{\mu+1}$

Look up characteristic equations & Julia's Office!
non-homogeneous equations

Quiz
Up to 6.5

§7.6 Perturbation Theory

$$\underbrace{\ddot{x} + x}_{\substack{\text{SHO}\\ \text{(unperturbed)}}} + \underbrace{\varepsilon h(x, \dot{x})}_{\substack{\text{perturbation}\\ \text{(nonlinear)}}} = 0, \quad 0 \le \varepsilon << 1$$

$h(x,\dot{x}) = (x^2-1)\dot{x} \leftarrow$ Van der Pol
$h(x,\dot{x}) = (x^3) \leftarrow$ Duffing

Avg 42
SD 6.5
Max 83

P.T. finds the solution to nonlinear perturbed equations using the unperturbed solution and corrections to it.

Assume:

(1) $x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$

x_0: soln to unperturbed

Ex) $\ddot{x} + 2\varepsilon\dot{x} + x = 0$, $x(0) = 0$, $\dot{x}(0) = 1$

by (1), $[\ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + \cdots] + [x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots] + 2\varepsilon[\dot{x}_0 + \varepsilon\dot{x}_1 + \varepsilon^2\dot{x}_2 + \dots] = 0$

$\mathcal{O}(1)$: $\ddot{x}_0 + x_0 = 0$
$\mathcal{O}(\varepsilon)$: $\ddot{x}_1 + x_1 = -2\dot{x}_0$
$\mathcal{O}(\varepsilon^2)$: $\ddot{x}_2 + x_2 = -2\dot{x}_1$

Exact Solution:
$x(t,\varepsilon) = (1-\varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin\left[(1-\varepsilon^2)^{1/2} t\right]$

$\mathcal{O}(1)$: $x_0(t) = A\sin t + B\cos t$
$\mathcal{O}(\varepsilon)$: $\ddot{x}_1 + x_1 = -2A\cos t + 2B\sin(t)$ / homogeneous: $x_{1h}(t) = C\sin t + D\cos t$
since $\omega_0 = \omega_1 = 1$, we have resonance

...resonance causes problems.

$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) \Rightarrow x(0) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)$
with $x_0 \to 0$, $x_1 \to 0$, $x_2 \to 0$

$\boxed{x_0(t) = \sin t}$, $\dot{x}_0(t) = \cos t \leftarrow$

$\dot{x}(0) = 1 = \dot{x}_0(t) + \varepsilon\dot{x}_1(t) + \varepsilon^2\dot{x}_2(t)$
with $\dot{x}_0 \to 1$, $\dot{x}_1 \to 0$, $\dot{x}_2 \to 0$

$\ddot{x}_1 + x_1 = -2\cos t$ (look this up)
$x_1(t) = -t\sin t \leftarrow$ secular term $\leftarrow$ approximation breaks when $t\varepsilon \not< 1$

Problems:

Exact solutions have 2 timescales: $\mathcal{O}(1)$ fast times (sin); $\mathcal{O}(1/\varepsilon)$ damping (exp)
Oscillation frequency is not quite 1, but $(1-\varepsilon^2)$

To treat secular terms:

let $\underset{fast}{\tau} \equiv t$, $\underset{slow}{T} \equiv \varepsilon t$, considered independent. "shove it in then crank it"

$$x(t,\varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \varepsilon^2 x_2(\tau, T)$$

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial x}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial x}{\partial \tau} + \varepsilon \frac{\partial x}{\partial T}$$

$$\rightarrow \dot{x} = \partial_\tau x + \varepsilon \partial_T x + \mathcal{O}(\varepsilon^2) = \partial_\tau x_0 + \varepsilon(\partial_T x_0 + \partial_\tau x_1) + \mathcal{O}(\varepsilon^2)$$

$$\ddot{x} = \partial_{\tau\tau} x + \varepsilon(\partial_{\tau\tau} x_1 + 2\partial_{\tau T} x_0) + \mathcal{O}(\varepsilon^2)$$

(cont.)

IDEA: $\underbrace{\mathcal{L}(x,\dot{x},\ddot{x})}_{\text{easily solvable}} + \varepsilon \underbrace{H(x,\dot{x},\ddot{x})}_{\text{small perturbations}} = 0$, $0 \le \varepsilon << 1$

Let $\lambda_{(n)}$ be eigenvalues of $A^{n \times n}$, $N_{(n)}$ be for $B^{n \times n}$

The eigenvalues for $(A+B)$ are not easily known (except when they can be triangularized). $Tr(A+B) = Tr(A) + Tr(B)$, which establishes some boundaries.

$$\sum \lambda_{a,b} = \sum \lambda_a + \sum \lambda_b$$

Matrices: $H_0 + \varepsilon V$

$$\left.\begin{array}{l} \underset{\substack{\uparrow \text{known} \\ \lambda_a, \vec{v}_a}}{A} + \varepsilon B \end{array}\right\} \lambda_{a+\varepsilon B} \text{ can be found}$$

Perturbation: $\frac{\varepsilon \|B\|}{\|A\|} << 1$

$V^T V$

$A^T A$

Strategy: $\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$

① Assume $x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) \ldots$

② Plug in, collect terms by powers of ε.

$\mathcal{O}(\varepsilon^0) \rightarrow \ddot{x}_0 + x_0 = 0 \rightarrow x_0(t) = A\sin t + B\cos t$

$\mathcal{O}(\varepsilon^1) \rightarrow \ddot{x}_1 + x_1 = -h(x_0, \dot{x}_0) = -2A\sin t + 2B\sin t$

$\mathcal{O}(\varepsilon^2) \rightarrow$

(cont.)

$\tau \equiv t$, $T \equiv \varepsilon t$ Ex| $\underbrace{\ddot{x} + x}_{\mathcal{L}(x,\dot{x},\ddot{x})} + \underbrace{2\varepsilon\dot{x}}_{\varepsilon H(x,\dot{x},\ddot{x})} = 0$

(fast) (slow)

Assume ① $x(\tau,T) = x_0(\tau,T) + \varepsilon x_1(\tau,T) + \varepsilon^2 x_2(\tau,T)$

$\dot{x}(\tau,T) = \partial_\tau x + \varepsilon \partial_T x$

② $\dot{x}(\tau,T) = \partial_\tau x_0 + \varepsilon(\partial_\tau x_1 + \partial_T x_0) + \mathcal{O}(\varepsilon^2)$

③ $\ddot{x}(\tau,T) = \partial_{\tau\tau} x_0 + \varepsilon(\partial_{\tau\tau} x_1 + 2\partial_{\tau T} x_0) + \mathcal{O}(\varepsilon^2)$

$\ddot{x} + x + 2\varepsilon\dot{x} = 0$

$\cancel{\partial_{\tau\tau} x_0} + \varepsilon(\cancel{\partial_{\tau\tau}} x_1 + 2\partial_{\tau T} x_0) + \cancel{x_0} + \varepsilon x_1 + 2\varepsilon \partial_\tau x_0 + \mathcal{O}(\varepsilon^2)$

$(\partial_{\tau\tau} x_0 + \cancel{x_0}) + \varepsilon(\partial_{\tau\tau} x_1 + 2\partial_{\tau T} x_0 + \cancel{x_1} + 2\,\partial_\tau x_0) + \mathcal{O}(\varepsilon^2) = 0$

$\mathcal{O}(1)$: $\partial_{\tau\tau} x_0 + x_0 = 0 \rightarrow x_0 = A(T)\sin\tau + B(T)\cos\tau$

$\mathcal{O}(\varepsilon)$: $\underbrace{\partial_{\tau\tau} x_1 + x_1}_{} = 2\partial_{\tau T} x_0 + 2\partial_\tau x_0$
$= -2(A' + A)\cos\tau + 2(B' + B)\sin\tau$

homogeneous soln.

$x_1(\tau) = C\sin\tau + D\cos\tau$

To remove resonance,

$\partial_T A + A = 0 \Rightarrow A(T) = A(0)e^{-T}$

$\partial_T B + B = 0 \quad B(T) = B(0)e^{-T}$

$\therefore x_0(\tau,T) = A(0)e^{-T}\sin\tau + B(0)e^{-T}\cos\tau$

$x_0(\tau,T) = 0$; $B_0 = 0$

$x_0(\tau,T) = A(0)e^{-T}\sin\tau$

$\partial_\tau x_0(0,0) = A(0) = 1$

$x(\tau,T) \approx e^{-T}\sin\tau$; $x(t) = e^{-\varepsilon t}\sin t$

All in all, pretty good.

∂

$\partial\, 2$

§9,10 Chaos Theory

Ed $\boxed{\text{Lorenz}}$ in $\boxed{1963}$ modelled the weather on a smooth sphere.

Strange Attractor
$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

σ, r, b are parameters (look up paper)

Features — high sensitivity to initial conditions.
(traj. that start near diverge exponentially)

Let $\vec{x}(t)$ be the traj. at $\vec{x}_0$

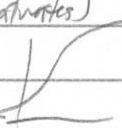
$\vec{x}(t) + \delta(t)$ from $\vec{x}(0) + \vec{\delta}(0)$

$|\vec{\delta}(0)| \sim 10^{-15}$

Simplest:
Rössler attractor
(only 1 non-linear term)

$\delta(t)$ quantifies the divergence of two initially nearby trajectories.

Numerically $\|\vec{\delta}(t)\| = \|\vec{\delta}_0\| e^{\lambda t}$ $(\lambda = 0.9)$ (after large time, saturates)



$\lambda > 0$ is a signature of chaos — Lyapunov Exponent

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \underbrace{\left\| \frac{\vec{\delta}(t)}{\vec{\delta}(0)} \right\|}_{\text{ratio} > 1 \text{ implies chaos}}$$

Defn of Chaos

Aperiodic long-term behaviour in a deterministic system that exhibits sensitivity to initial conditions.

never settles to a fixed point, limit cycle, or periodicity.

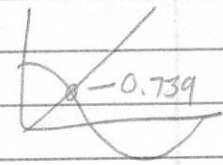
no random input, can write governing equations in absolute terms.

exponential divergence of nearby solutions

§10 1D Maps

Study difference equations of the form $x_{n+1} = f(x_n)$: maps

Ex $x_{n+1} = \cos x_n$; $x_0 = 2 \to x = 0.739$



$\Delta x = f(x_n) - x_n$

Uses:

1. numeric computation
2. discrete models of digital electronics, econ, finance
3. simplest examples of chaos

x^* is a f.p. when $x^* = f(x^*)$

Stability: $x^* + \eta_{n+1} = \underbrace{f(x^* + \eta_n)}_{\text{taylor expansion at } x^*}$, $|\eta_0| << 1$

$$x^* + \eta_{n+1} = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2)$$

$$\eta_{n+1} = f'(x^*)\eta_n \longrightarrow \eta_n = [f'(x^*)]^n \eta_0$$

$\lambda \equiv f'(x^*) \to \eta_n = \lambda^n \eta_0$. For $|\lambda| < 1$, stable
$|\lambda| > 1$, unstable
$\lambda = 1$, check $O(\eta^2)$ ...

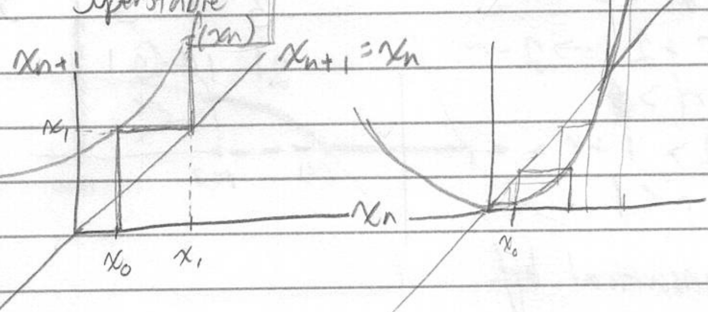
Cobweb Diagrams: Graphical insight into global stability

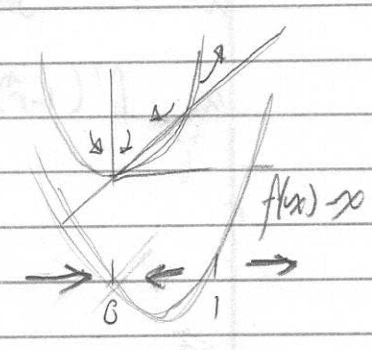
Ex $x_{n+1} = x_n^2 \to f(x) = x^2$, $f'(x) = 2x$

f.p. – $x^* = \{0, 1\}$

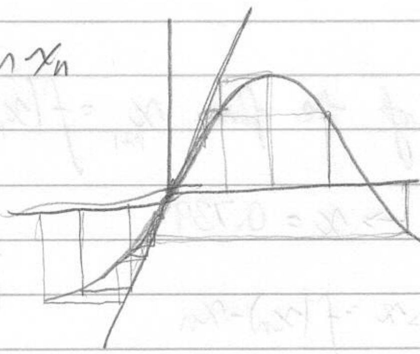
$x^* = 0$: $\lambda = 0$ Stable $x^* = 1$: $\lambda = 2$: Unstable

"Superstable"

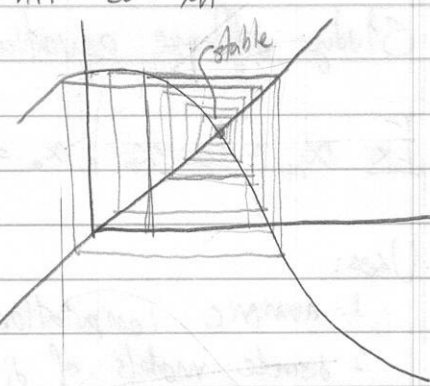




$x_{n+1} = \sin x_n$



$x_{n+1} = \cos x_n$



S. Strogatz
M. Feigenbaum
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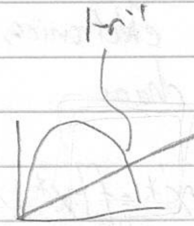
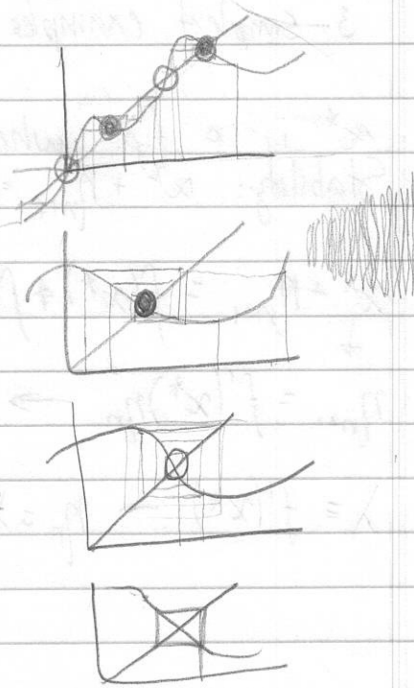
M. Mandelbrot
J. Gleick





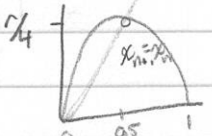
Logistic Map

★ $x_{n+1} = r x_n (1 - x_n)$

$x^* = r x^* (1 - x^*)$ $\quad f'(x^*) = r - 2r x^*$

$r^{-1} = 1 - x^*$

$x^* = 1 - r^{-1}$



$x^* = 0$: stable for $|r| < 1$

unst. for $|r| > 1$

$x^* = 1 - r^{-1}$: stable for $|r| < 1$

$f'(1 - r^{-1}) = r - 2r(1 - r^{-1})$

$= r - 2r + 2$

$-r + 2 \to 2 - r$

$|2 - r| > 0$

$2 > 1 + r$

$r < 1$

$r \to [0, 3] \to$ transcritical bif.

$x_{n+1} =$

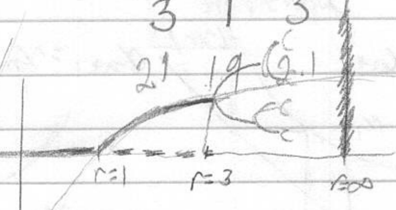
$f'(2) = -1$

$f(2) = 2$

$f = -x + 4$

3 1 3 | 3 1

2.1 1.9 2.1



Sine Map

$x_{n+1} = r\sin(\pi x_n)$

Exhibits same properties of chaos as logistic map.

Unimodal Map

Simple smooth f w/ only one maximum; concave down.

Thm

$x_{n+1} = rf(x_n)$; $f(0) = f(1) = 0$, unimodal.

As r is varied, the order in which stable periodic orbits appear is independent of the unimodal map.

Feigenbaum showed that
- r_n converges: $r_\infty < \infty$
- distance between transitions shrinks by 4.669

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669$$

universal among unimodal maps where $x_{n+1} = rf(x_n)$; $f(0) = f(1) = 0$